London Mathematical Society Lecture Note Series 84

Finite Group Algebras and their Modules

P. LANDROCK

CAMBRIDGE UNIVERSITY PRESS

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London Mathematical Society Lecture Note Series: 84

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CAMBRIDGE UNIVERSITY PRESS
Cambridge
London New York New Rochelle
Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

First published 1983

Printed in Great Britain at the University Press, Cambridge

Library of Congress catalogue card number: 83-15049

British Library Cataloguing in Publication Data

Landrock, P.
Finite group algebras and their modules -
(London Mathematical Society lecture note series,
ISSN 0076-0552; 84)
1. Finite groups
I. Title II. Series
512'.2 QA171

ISBN 0 521 27487 7

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PREFACE

This book is concerned with the structure of group algebras of finite groups over fields of characteristic p dividing the order of the group, or closely related rings such as rings of algebraic integers and in particular their p-adic completions, as well as modules, and homomorphisms between them, of such group algebras.

Our principal aim has been to present some of the more recent ideas which have enriched and improved this beautiful theory that owes so much to Richard Brauer. In other words, we wish to account for a major part of what could be described as the post-Brauer period. The reader will find that once we get started, the majority of our proofs have not appeared before in any textbooks, and as far as Chapters II and III are concerned, a number of results and proofs which have not appeared before at all are included.

We do not at any stage restrict ourselves to particular methods, be they ring theoretic, character theoretic, etc. In each case we have attempted to present a proof or an approach which distinguishes itself in one way or another perhaps by being fast, elegant, illuminating, or with promising potentials for further advancement, or possibly all of this at the same time. (We are well aware of the fact that the reader may not always agree this has been achieved (unless of course he or she recognizes his or her own proof!)) One point though that has been important to us is to demonstrate the strong connection to cohomology which undoubtedly will be strengthened in the years to come. Another point to make is that we have tried very hard to avoid assumptions on the coefficient rings involved in the ambitious hope to attract non-specialists, perhaps even algebraic topologists and group theorists who may feel tempted to use the tools of modular representations more frequently.

Of course, to make the presentation as smooth, coherent and self-contained as possible, many classical results are included. Thus we only require knowledge with the theory of semisimple algebras and modules, including basic character theory (if this is not present, we recommend Feit (1967), Serre (1967) or Isaacs (1976)) and elementary facts about finite groups. Also to advance to the frontier as quickly as possible we have added suitable hypotheses at an early stage whenever convenient if it saves us some time. Just as an example, we only prove Krull-Schmidt for finite-dimensional algebras, not artinian rings in general. Usually, we will give a reference to Curtis and Reiner (1981) and (1985) for the more general results.

As the whole idea is to present--whenever appropriate--methods that Brauer avoided or did not even have at hand, the reader will find relatively few references to Brauer's work with the exception of more recent papers such as (1968), (1969) and (1971), and Brauer and Feit (1959). As references to Brauer's Main Theorems, we use the survey articles (1956) and (1959) rather than the original papers and otherwise refer to Feit (1982), which gives a very detailed account of Brauer's work and methods. The justification is that if we want to improve Brauer's theory substantially, we have to come up with something completely new. Recent contributions to which we have devoted particular attention are among others Alperin and Broué (1979), Benson and Parker (1983), Brauer (1968) and (1971), Brandt (1982b), Burry and Carlson (1982), Feit (1969), Green (1974), Knörr (1979), Landrock (1981c) and Scott (1973). This choice is no indication of an attempted evaluation of importance. These are simply the sources we have decided in particular to work with or discuss, leaving others out which equally well deserve careful attention such as Dade's deep work on endopermutation modules or Puig (1981) which is very far-reaching, as well as a number of other topics. Also we do not concern ourselves with the theory of blocks with cyclic defect groups, nor with p-solvable groups, which have recently been treated with great care and detail in Feit (1982). Likewise, Glauberman's powerful and important Z^*-Theorem, which has been indespensable in the classification of the finite simple groups has not been included for the simple reason that we have nothing new to contribute which is not already treated in the literature (see Feit (1982) again, for instance).

It is striking however how many of the deeper results in block theory were anticipated by R. Brauer and how hard we have to work to advance further. And we want to point out that some of Brauer's later work (quoted above) has been a major source of inspiration to a number of people over the last decade, which is the reason why a major part of Chapter III, Section 8, is devoted to these papers.

In 1971, Paul Fong gave a well-composed and inspiring course at Aarhus University on representation theory (see Fong (1971)), which in turn was partly inspired by Dade's lecture notes (1971) and Green's fundamental work in the sixties. Since 1975, I have given a number of lecture series at Aarhus on this subject, which gradually have developed from being close to Fong's lecture notes into part of the present book. Other direct or indirect sources of inspiration have been Michler (1972) and in particular Green (1974) apart from a great deal of Green's work in the sixties and seventies, which perhaps is the major general source of inspiration for Chapter II. Also I have profitted a lot from useful comments by and discussions with my students and others who have attended my lectures, in particular, Ivan Damgård and Carsten Hansen from the first category and Dave Benson from the second, all of whom helped me avoiding considerably more blunders than present now. Other results or approaches are inspired from my collaboration with G. Michler and discussions with J. L. Alperin, H. Jacobinski and D. Sibley and I have enjoyed comments from K. Fuller who read part of Chapter I and D. Burry who read part of Chapter II. The first version of Chapter II similar to the first half of the present was conceived and presented during my visit at University of Oxford in the spring of 1981 and I want to thank Michael Collins warmly for making this possible, and the British Science and Engineering Council for its financial support. But the major part of the final version was written during the academic year 1982-83 at the Institute for Advanced Study, Princeton. I am extremely grateful to the Institute, to NSF Grant MCS-8108814 (A01), and to Aarhus University for the help and financial basis for my stay there. And I want to thank Marianne for her support and understanding as well.

Finally, I wish to thank Peggy Murray (who typed Chapter II) and in particular Kathy Lunetta (who typed the rest) for their excellent, fast and reliable typing as well as their patience with me and my manuscript.

A few remarks on notation and basic assumptions: If A is a ring, A_A means A considered as a right A-module, and except for a very few cases, a module is always right and finitely generated, free over the ring of coefficients. Also, if G is a group and $X \subseteq G$, $a \underset{G}{\in} X$ means $a^g = g^{-1}ag \in X$ for some $g \in G$. Likewise if $H, K \leq G$, $H \underset{G}{\leq} K$ means

$H^g \leq K$ for some $g \in G$. Furthermore, H/G means an arbitrary right transversal of H in G, $G\backslash H$ an arbitrary left and $H\backslash G/K$ an arbitrary transversal of double coset representatives.

One more thing: As one tool is used over and over again, it only seems fair to express our gratitude towards this as well. Therefore in more than one sense of the work, this book is dedicated to the trace map.

Princeton, New Jersey
June, 1983
Peter Landrock

1. Idempotents in rings. Liftings.

In this section, A is an arbitrary ring. Recall that an element $0 \neq e \in A$ is called an idempotent if $e^2 = e$. Two idempotents e_1 and e_2 are said to be orthogonal if $e_1e_2 = e_2e_1 = 0$, and an idempotent is called primitive if it is not the sum of two orthogonal idempotents.

Definition 1.1. Let A be a ring and M an A-module. Then M is said to be decomposable if there exist non-trivial submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$. Otherwise, M is called indecomposable.

Lemma 1.2. Let A be a ring and $e \in A$ an idempotent. Then eA is indecomposable as an A-module if and only if e is primitive.

Proof: One way is trivial. Conversely, assume $eA = A_1 \oplus A_2$ where $0 \neq A_i$ is a right ideal for $i = 1,2$. In particular, $e = e_1 + e_2$ for some $e_i \in A_i$, $i = 1,2$. Moreover, $ee_i = e_i$ for $i = 1,2$, as $e_i \in eA$. Hence

(1) $$e_1e_2 = (e-e_2)e_2 = e_2 - e_2^2 \in A_1 \cap A_2 = 0.$$

Thus $e_1e_2 = 0$ and $e_2^2 = e_2$. So by symmetry, $e_2e_1 = 0$ and $e_1^2 = e_1$ as well. Thus e is not primitive.

Definition 1.3. By an idempotent decomposition of 1 in A, we understand a set of pairwise orthogonal idempotents $e_1,\ldots,e_r$ such that $1 = \sum_{i=1}^{r} e_i$. An idempotent decomposition is called primitive if all the involved idempotents are primitive.

Remark. The importance of idempotent decompositions is of course, in view of Lemma 1.2, that they correspond to direct sum decompositions. If $A_A = \bigoplus_{i=1}^{t} A_i$, then $1 = \sum_{i=1}^{t} e_i$, where $e_i \in A_i$, is necessarily an idempotent decomposition, and vica verca. But even more holds, namely

Theorem 1.4 (Fitting). Let A be a ring and M an A-module. Denote the endomorphism ring of M over A by E. Then

i) There is a one-to-one correspondence between idempotent decompositions $1 = \sum_{i \in I} e_i$ in E and decompositions $M = \bigoplus_{i \in I} M_i$, where I is finite, characterized by the fact that e_j is the projection of M onto M_j with kernel $\bigoplus_{i \neq j} M_i$.

ii) Let $M = M_1 \oplus M_2 = N_1 \oplus N_2$, and let e be the projection onto M_1 with kernel M_2, f the projection onto N_1 with kernel N_2. Then $M_1 \simeq N_1$ if and only if $eE \simeq fE$ as E-modules.

iii) Let $e \in E$ be an idempotent. Then $e(M)$ is indecomposable if and only if eE is indecomposable.

Proof: i) and iii) are obvious.

ii) Let $\phi : M_1 \to N_1$ be an isomorphism. We may consider ϕ as an element of E by setting $\phi(M_2) = 0$. Then $\phi = f\phi e$. We therefore define $\Phi : eE \to fE$ by $\Phi(\alpha) = \phi\alpha$. As ϕ is an isomorphism, it easily follows that Φ is as well. Conversely, let $\Phi : eE \to fE$ be an isomorphism of E-modules. Let $\Phi(e) = f\phi_f$, $\Phi(e\phi_e) = f$, where $\phi_f, \phi_e \in E$. Then $f\phi_f e = \Phi(e)e = \Phi(e) = f\phi_f$. Similarly, $e\phi_e f = e\phi_e$, as $\Phi(e\phi_e f) = \Phi(e\phi_e)f = f = \Phi(e\phi_e)$. Also, $f = \Phi(e\phi_e) = \Phi(e)\phi_e = f\phi_f\phi_e$. Similarly, $e = e\phi_e\phi_f$, as $\Phi(e\phi_e\phi_f) = \Phi(e\phi_e)\phi_f = f\phi_f = \Phi(e)$. But then

(2) $$(f\phi_f)(e\phi_e) = f\phi_f\phi_e = f$$

(3) $$(e\phi_e)(f\phi_f) = e\phi_e\phi_f = e$$

which proves that $f\phi_f : M_1 \to N_1$ is an isomorphism.

We end this section with a very important theorem on lifting idempotents. Recall that an element $v \in A$ is called nilpotent if there exists $n \in \mathbb{N}$ such that $v^n = 0$. If v is nilpotent, then obviously $1 + v$ is a unit.

<u>Theorem 1.5</u>. Let A be a ring and N a nilpotent ideal of A. Then

i) Let $\bar{e}$ be an idempotent of $\bar{A} = A/N$. Then there exists an idempotent e in A such that $e + N = \bar{e}$; ($\bar{e}$ is said to be <u>lifted</u> to e). If e' is another such idempotent, there exists $v \in N$ such that

$$e' = (1+v)^{-1} e(1+v). \tag{4}$$

ii) Units of $\bar{A}$ always lift to units of A.

iii) Let $\bar{1} = \sum_{i=1}^{t} \bar{e}_i$ be an idempotent decomposition in $\bar{A}$. Then there exists an idempotent decomposition $1 = \sum_{i=1}^{t} e_i$ in A such that $\bar{e}_i = e_i + N$ for all i. Again, if $\sum_{i=1}^{t} e_i'$ is another such decomposition in A, there exists $v \in N$ such that

$$e_i' = (1+v)^{-1} e_i(1+v) \tag{5}$$

for all i.

iv) Let $\bar{e} \in \bar{A}$ be an idempotent and let $e \in A$ be an idempotent such that $e + N = \bar{e}$. Then $\bar{e}$ is primitive if and only if e is primitive.

<u>Proof</u>: We first prove the theorem under the additional assumption that $N^2 = 0$.

i) Let $f \in A$ such that $f + N = \bar{e}$. Then $f^2 = f + y$ for some $y \in N$. Furthermore, for any $x \in N$,

$$(f+x)^2 = f^2 + xf + fx = (f+x) - x + y + xf + fx. \tag{6}$$

Thus we want to choose x such that $y = x - xf - fx$. To obtain this, we magically choose $x = (1-2f)y$. As $y = f^2 - f$, x commutes with f and

(7) $$x - xf - fx = (1-2f)y - 2f(1-2f)y$$
$$= (1 + 4f^2 - 4f)y$$
$$= (1+4y)y = y$$

as $y \in N$. Thus $e = f + (1-2f)(f^2-f)$ indeed is an idempotent in $\bar{e}$.

Next let e' be another idempotent in $\bar{e}$. Hence $e' = e + z$ for some $z \in N$. Then $e + z = e + ez + ze$ and thus $(1-e)z = ze$, which forces $eze = 0$. Likewise, $(1-e)z(1-e) = 0$. But now, for any $r \in A$,

(8) $$r = ere + er(1-e) + (1-e)re + (1-e)r(1-e).$$

Thus (8) reduces to

(9) $$z = ez(1-e) + (1-e)ze$$

for $r = z$. To finish, we need $v \in N$ such that

(10) $$e + z = (1-v)e(1+v) = e - ve + ev$$

since $vev \in N^2 = 0$. This forces $z = ev - ve$. So this time, we define

(11) $$v := ez(1-e) - (1-e)ze$$

which has the required property by (9).

ii) Let $u + N = \bar{u}$ for $\bar{u}$ a unit in $\bar{A}$. Then there exists $v \in A$ such that $\bar{u}\bar{v} = \bar{v}\bar{u} = \bar{1}$ with $\bar{v} = v + N$. Thus $uv = 1 + y$ and $vu = 1 + z$ for suitable $y,z \in N$. Hence uv and vu are units, which in turn forces u to be a unit.

iii) We use induction on t, the first step being i). Furthermore, by i) again, there exists an idempotent $e_t \in A$ such that $e_t + N = \bar{e}_t$. Let $A' = (1-e_t)A(1-e_t)$ which is a subring of A with $1 - e_t$ as unity. Moreover, $e_t r = re_t = 0$ for all $r \in A'$. The homomorphism $A \to \bar{A}$ induces a homomorphism of A' onto $(\bar{1}-\bar{e}_t)\bar{A}(\bar{1}-\bar{e}_t)$ with kernel $N' = A' \cap N$. In particular, $N'^2 = 0$. However, $\bar{e}_1,\ldots,\bar{e}_{t-1}$ all lie in $(\bar{1}-\bar{e}_t)\bar{A}(\bar{1}-\bar{e}_t)$, and $\bar{1}-\bar{e} = \sum_{i=1}^{t-1} \bar{e}_i$ is an idempotent decomposition in this ring. Hence induction yields the existence of an idempotent decomposition $1 - e_t = \sum_{i=1}^{t-1} e_i$ in A', thus proving the first part of iii).

To show uniqueness in the sense as stated in iii), we again apply induction on t and as before, i) establishes the case $t=1$. Moreover, i) allows us to assume in the general case that $e_t = e'_t$. Induction now yields a $v \in N' \subseteq N$ such that $(1-v)e'_i(1+v) = e_i$ for all $i \leq t-1$. But as $v \in A'$, $e_t v = v e_t = 0$. Hence $(1-v)e_{t'}(1+v) = e_{t'} = e_t$ as well.

iv) One way is trivial. Conversely, assume $e = \bar{f}_1 + \bar{f}_2$ with $\bar{f}_1$ and $\bar{f}_2$ orthogonal idempotents. Now e is the unity of eAe and $A \to \bar{A}$ induces a homomorphism of eAe onto $\overline{eAe}$ with kernel $N \cap eAe$. Then i) asserts that $\bar{e} = \bar{f}_1 + \bar{f}_2$ can be lifted to eAe, and e is not primitive.

Finally, if N is an arbitrary nilpotent ideal with $N^r = 0$, we first lift to A/N^2 and apply induction on N^i to lift to $A/N^r = A$.

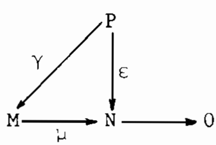
2. <u>Projective and injective modules</u>.

For the convenience of the reader, we recall the basic properties of projective and injective modules.

Let A be a ring. Then the direct summands of A_A have particularly nice properties. One of them is the property defined in

<u>Definition 2.1</u>. Let A be a ring. Then an A-module P is called <u>projective</u> if for any two A-modules M and N, and A-homomorphisms $\mu : M \to N$, which is surjective, and $\varepsilon : P \to N$, there exists a homomorphism $\gamma : P \to M$ such that

(1)



commutes.

<u>Theorem 2.2</u>. Let A be a ring, P_1, P_2 and P A-modules. Then we have

i) Any free A-module is projective. In particular, A_A is a projective A-module.

ii) $P_1 \oplus P_2$ is projective if and only if P_i is projective, $i = 1,2$.

iii) P is projective if and only if there exists an A-module M such that $P \oplus M$ is free.

iv) For any A-module M, there exists an exact sequence $0 \to N \to P_M \to M \to 0$ with P_M projective.

P is projective if and only if every exact sequence $0 \to N \to M \to P \to 0$ splits.

Proof: The reader is probably already familiar with these homological trivialities. Otherwise he or she is urged to produce the proofs.

Having defined projectivity one may feel tempted to discuss the dual property, injectivity. We shall see later that for group algebras they are identical. Nevertheless, it is quite convenient to be aware of the formal difference. Moreover, if we turn to algebraic groups over infinite fields, there really is a difference.

Definition 2.3. An A-module I is called injective if for any two A-modules M and N, and A-homomorphisms $\lambda : N \to M$, which is injective, and $\varepsilon : N \to I$, there exists a homomorphism $\rho : M \to I$ such that the following diagram commutes

(2)

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\lambda} & M \\ & & {\scriptstyle\varepsilon}\downarrow & \swarrow{\scriptstyle\rho} & \\ & & I & & \end{array}$$

Theorem 2.4. Let I_1, I_2 and I be A-modules. Then

i) $I_1 \oplus I_2$ is injective if and only if I_i is injective, $i = 1,2$.

ii) For any module N there exists an exact sequence $0 \to N \to I_N \to M \to 0$ with I_N injective.

iii) I is injective if and only if every exact sequence $0 \to I \to M \to N \to 0$ splits.

Proof: See the proof of Theorem 2.2. (ii) needs some elaboration.)

We will in the following use these basic properties of projective and injective modules without giving special reference, as a main rule.

3. The radical and artinian rings.

A discussion of the following definitions and basic results may be found in a great number of books on ring theory, of which Artin, Nesbitt and Thrall (1944) is the classical source. For a more contemporary treatment which is in concurrence with our discussion here, we suggest Anderson and Fuller (1973). Anyway, before embarking on the study of this section, the reader should make sure to be familiar with the theory of semisimple rings and modules.

Definition 3.1. Let A be a ring. The radical of A, which will be denoted by $J(A)$, is defined as the intersection of all maximal right ideals of A.

Lemma 3.2. Let A be a ring. Then

i) Let E be a simple A-module. Then $EJ(A) = 0$.

ii) Let $x \in A$ and assume $Ex = 0$ for all simple A-modules E. Then $x \in J(A)$.

iii) $J(A)$ is a 2-sided ideal.

Proof: Let E be an arbitrary but fixed simple A-module, and choose $v \in E$ with $vA \neq 0$. Then $a \to va$ defines a module homomorphism $A \to E$. Denote the kernel of this map by M_E. As E is simple, M_E is a right maximal ideal in A, and thus contains $J(A)$ by definition. This proves i) and ii), which together characterize $J(A)$ as the set of elements in A which annihilates all simple A-modules, from which iii) follows.

Definition 3.3. Let A be a ring, M and A-module. Then M is called (right) artinian, if any descending chain of submodules becomes stationary at some point. M is also said to satisfy the d.c.c. (descending chain condition), or the minimal condition.

Likewise, A is called a right (left) artinian ring, if A_A $({}_AA)$ is artinian. If A is right and left artinian, A is called artinian.

Definition 3.4. A descending chain

(1) $$M = M_0 \supset M_1 \supset \dots \supset M_n = 0$$

of submodules of the A-module M is called a composition series if M_i/M_{i+1} is simple for all i.

Remark. If M is not finitely generated, M may not have a composition series, even if A is right artinian. In fact, if M is not finitely generated, M may not even have a maximal submodule. However, if M is finitely generated, there is no problem as we proceed to see. The point is

Lemma 3.5. Let A be a ring, and let M be a finitely generated A-module. Then M has a maximal submodule. In particular, $MJ(A) \subset M$.

Proof: The reader is probably familiar with the fact that the first statement follows from Zorn's lemma. Now the second follows from Lemma 3.2i).

Theorem 3.6. Let A be a right artinian ring. Then

i) $J(A)$ is nilpotent.

ii) Let M be a finitely generated A-module. Then $M/MJ(A)$ is semisimple.

iii) $A/J(A)$ is a semisimple ring.

Proof: i) If $J(A)$ is not nilpotent, there exists an $n \in \mathbb{N}$ with $J(A)^n = J(A)^{n+1} \neq 0$ by definition. Hence there exists $a \in J(A)$

with $aJ(A)^n \neq 0$. Moreover, as A is artinian, we may even assume that $I = aA$, I a minimal ideal such that $IJ(A)^n \neq 0$. Thus in fact $aA = aJ(A)$ by minimality of I, as $aJ(A)^{n+1} = aJ(A)^n$, and thus $aAJ(A) = aA$, a contradiction by Lemma 3.5.

To prove ii) and iii), it suffices to prove that $A/J(A)$ is a semisimple A-module. But as A is artinian, there exists finitely many right maximal ideals $M_1,\ldots,M_r$ such that $J(A) = \bigcap_{i=1}^{r} M_i$ by the characterization of $J(A)$ in the proof of Lemma 3.2. Hence the canonical homomorphism $A/J(A) \to \bigoplus_{i=1}^{r} A/M_i$ is injective.

Corollary 3.7. Let A be a right artinian ring and M a finitely generated A-module. Then $MJ(A)$ is the unique minimal submodule with $M/MJ(A)$ semisimple.

Notation: In view of Corollary 3.6, we set $J(M) := MJ(A)$ and call this the radical of M.

Corollary 3.8. Let A be a right artinian ring and M a finitely generated A-module. Then

i) M is isomorphic to a direct sum of indecomposable modules.

ii) M has a composition series.

Proof: This follows immediately from the fact that $J(A)$ is nilpotent and $A/J(A)$ and $M/MJ(A)$ are semisimple, as does

Corollary 3.9 (Nakayama's lemma). Same assumptions as above. Let L be a submodule of M such that $L + MJ(A) = M$. Then in fact $L = M$.

The following is now straightforward though tedious to establish using the fundamental homomorphism theorem and we (wisely) omit the proof.

Theorem 3.10 (Jordan-Hölder). Let A be a ring and M an A-module. If M possesses a composition series, any two composition series contain the same number of members, and the simple factor modules arising from these series may be arranged to be pairwise isomorphic.

Definition 3.11. Let A be a ring, and assume $A_A = \bigoplus_{i=1}^{t} P_i$, where P_i is indecomposable. (This, for instance, holds if A is right artinian, as we have just seen.) These summands are called the principal indecomposable modules (p.i.m.'s) of A.

It is now clear from Lemma 1.2 that a right ideal P in a ring A, with the above property, is a p.i.m. if and only if there exists a primitive idempotent e such that P = eA.

Before we prove a number of important structure theorems describing the p.i.m.'s of a right artinian ring, we need the following important consequence of Theorem 1.5:

Theorem 3.12. Let A be a right artinian ring. Then

i) Let $1 = \sum_{i=1}^{t} e_i = \sum_{j=1}^{s} f_j$ be primitive idempotent decompositions. Then s = t and there exists a unit u in A such that $u^{-1}e_i j = f_{\phi(i)}$ for all i, where ϕ is some permutation of $\{1,2,\ldots,s\}$.

ii) Let e,f A be idempotents. Then $eA \simeq fA$ if and only if there exists a unit $u \in A$ such that $u^{-1}eu = f$.

Proof: We assume these results are familiar to the reader if A is semisimple. But then i) follows from Theorem 1.5 while for ii) we have to remark in addition that if $eA \sim fA$, then $eA/eJ(A) \simeq fA/fJ(A)$.

Corollary 3.13. Let A be a right artinian ring. Then

i) The p.i.m.'s of A_A are uniquely determined up to isomorphism. In other words, if

$$A_A \simeq \bigoplus_{i \in I} P_i \simeq \bigoplus_{j \in J} Q_j \tag{2}$$

where the P_i's and Q_j's are all indecomposable, then there exists a bijection $\phi : I \to J$ such that $P_i \simeq Q_{\phi(i)}$ for all i.

ii) A finitely generated indecomposable A-module is projective if and only if it is isomorphic to a p.i.m. of A_A.

Proof: i) is just a reformulation of Theorem 3.12i) in view of the remark following Definition 1.3.

ii) follows from the general properties of projective modules.

The following result is of extreme importance in what follows.

Theorem 3.14. Let A be a right artinian ring and $\{e_i\}$ a set of primitive idempotents. Set $P_i = e_iA$. Then

i) P_i contains a unique maximal submodule, namely $e_iJ(A)$.

ii) The following are equivalent

a) $P_i/e_iJ(A)$ and $P_j/e_jJ(A)$ are isomorphic

b) P_i and P_j are isomorphic

c) There exists a unit $\bar{u}$ in $A/J(A)$ such that $\bar{u}^{-1}\bar{e}_i\bar{u} = \bar{e}_j$, where $\bar{e}_k = e_k + J(A)$ for $k = i,j$

d) There exists a unit u in A such that $u^{-1}e_iu = e_j$.

Proof: i) Let M be a maximal right ideal in P_i. Then P_i/M is a simple A-module. In particular, $e_iJ(A) = P_iJ(A) \subset M$ by Lemma 3.2i). However, as $\bar{e}_i = e_i + J(A)$ is a primitive idempotent in $A/J(A)$ and $e_iA \cap J(A) = e_iJ(A)$, we have that $e_iA/e_iJ(A)$ is simple. Thus, in fact, $e_iJ(A) = M$.

ii) The equivalence of b) and d) was proved in Theorem 3.12, and c) and d) are equivalent by Theorem 1.5. Finally, the equivalence of c) and a) is a well-known property of semisimple rings.

In particular, we have proved

Corollary 3.15. There is a one-to-one correspondence between the isomorphism classes of the p.i.m.'s of A and the isomorphism classes of the simple A-modules.

4. Cartan invariants and blocks.

We proceed to define the so-called Cartan invariants of an arbitrary artinian ring.

Definition 4.1. Let $e_1,\ldots,e_m$ be primitive idempotents of the artinian ring A such that $\{e_iA\}$ form a complete set of representatives of isomorphism classes of p.i.m.'s of A (in particular, they are all orthogonal to each other). Let $P_i = e_iA$, and set $E_i = P_i/e_iJ(A)$.

The Cartan invariant c_{ij} is defined as the multiplicity of E_j as a composition factor in P_i. The $m \times m$ matrix $\{c_{ij}\} = C$ is called the Cartan matrix.

Later we shall prove several important results on Cartan invariants for group algebras. Here, in the general case where much less holds, we only prove

Lemma 4.2. The principal indecomposable P_i has a composition factor isomorphic to E_j if and only if P_ie_j $(= e_iAe_j) \neq 0$.

Proof: Assume $P_ie_j \neq 0$, and let $0 \neq x \in P_ie_j$. Then $xe_j = x$, and we may define an A-homomorphism $\phi : P_j \to P_ie_jA \subseteq P_i$ by $\phi(v) = xv$. Since P_j has a unique maximal submodule, namely $e_jJ(A)$, the kernel of ϕ must be contained in $e_jJ(A)$, and thus $\phi(P_j)/\phi(e_jJ(A)) \simeq E_j$ is a composition factor of P_i.

Conversely, if E_j is a composition factor of P_i, there exists a submodule M of P_i with a submodule N such that $M/N \simeq E_j$. As P_j is projective, the map $P_j \to E_j$ may be factored through M. In particular, there exists a non-trivial homomorphism $\phi : P_j \to P_i$. Hence $\phi(e_j) \neq 0$. But then $\phi(e_j) = \phi(e_j)e_j \neq 0$ which shows that $P_ie_j \neq 0$.

Definition 4.3. Let Q_1 and Q_2 be p.i.m.'s of A. Then Q_1 and Q_2 are said to be linked, if there exists a sequence of p.i.m.'s $Q_1 = P_1, P_2, \ldots, P_s = Q_2$ such that P_i and P_{i+1} have a composition factor in common for all i. For notation, we use $Q_1 \equiv Q_2$.

Clearly, $\equiv$ is an equivalence relation on the set of p.i.m.'s of A. Let $\mathbb{P}_1,\ldots,\mathbb{P}_r$ denote the equivalence classes under $\equiv$. By the block $\mathbb{B}_i$ of A associated with $\mathbb{P}_i$, we understand

(1) $$\mathbb{B}_i = \{\Sigma Q \mid Q \in \mathbb{P}_i,\ Q \subseteq A\}.$$

Theorem 4.4. The blocks of A are indecomposable 2-sided ideals of A and artinian rings. Moreover,

(2) $$A = \bigoplus_{i=1}^{r} \mathbb{B}_i$$

and (2) is the unique decomposition of A into a direct sum of indecomposable ideals. In particular, if e_i is the unity of $\mathbb{B}_i$, then $e_1,\ldots,e_r$ are the only centrally primitive idempotents in A.

Proof: By definition, $A = \Sigma\, \mathbb{B}_i$, and

(3) $$\mathbb{B}_i = \{\Sigma eA \mid eA \in \mathbb{P}_i,\ e \text{ a primitive idempotent}\}$$

which is a right ideal. Moreover, if $e \in \mathbb{B}_i$ and $f \in \mathbb{B}_j$ are primitive idempotents and $i \neq j$, Lemma 4.2 asserts that $eAf = 0$, by definition of blocks. Hence (3) yields that $\mathbb{B}_i \mathbb{B}_j = 0$. Consequently,

(4) $$A\mathbb{B}_i = (\Sigma\, \mathbb{B}_j)\mathbb{B}_i \subseteq \mathbb{B}_i \mathbb{B}_i \subseteq \mathbb{B}_i$$

and thus $\mathbb{B}_i$ is in fact a 2-sided ideal. Next we claim that the sum $\Sigma\mathbb{B}_i$ is direct. Indeed, this is a standard argument: Let $1 = \Sigma e_i$, where $e_i \in \mathbb{B}_i$, and let $0 = \Sigma a_i$, where $a_i \in \mathbb{B}_i$. Then

(5) $$a_j = a_j(\Sigma e_i) = a_j e_j = (\Sigma a_i)e_j = 0$$

for all j, as $\mathbb{B}_i\mathbb{B}_j = 0$ for $i \neq j$. It is now straightforward to show that e_i is a unity of $\mathbb{B}_i$ and that $\mathbb{B}_i$ is an artinian ring. Finally, let $A = \mathfrak{A} \oplus \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are 2-sided ideals and $\mathfrak{B}$ is indecomposable as such. Let $1 = e + f$ with $e \in \mathfrak{A}$ and $f \in \mathfrak{B}$. Again it follows that $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A} = 0$, and consequently f is a central idempotent, and primitive as such as $\mathfrak{B}$ is indecomposable. Hence there exists exactly one i with $e_i f \neq 0$. Thus $e_i = f$ as they are both primitive, and $\mathfrak{B} = \mathbb{B}_i$, from which the rest of the theorem follows.

We now leave the general theory of artinian rings to concentrate first on finite dimensional algebras and then group algebras and, to some extent, symmetric algebras.

5. Finite dimensional algebras.

First of all, we want to make the convention that for the rest of this book, a module is finitely generated.

In this particular section, we furthermore assume A to be a finite dimensional algebra over a field F. Obviously, A is an artinian ring then, and we will use the notation of Definition 4.1.

An important point we want to make in our whole discussion is how a number of crucial properties of a module M over A are closely related and often entirely determined by those of its endomorphism ring. We have already seen a demonstration of this in Theorem 1.4.

For M and N A-modules, we denote $\mathrm{Hom}_A(M,N)$ by $(M,N)^A$.

Lemma 5.1. With the notation above, we have

i) $(M,M)^A$ is a finite dimensional algebra over F.

ii) $J((M,M)^A) \supseteq \{\phi \in (M,M)^A \mid \phi(M) \subseteq MJ(A)\}$.

Proof: i) is trivial.

ii) Let $\mathfrak{n}$ denote the right hand side of ii), which obviously is an ideal. Let $\phi \in \mathfrak{n}^r$ for some r. Then $\phi(M) \subseteq MJ(A)^r$ and thus $\mathfrak{n}$ is nilpotent, which proves that $\mathfrak{n} \subseteq J((M,M)^A)$.

Remark. Equality does not always hold in ii) above.

Recall there is a one-to-one correspondence between idempotent decompositions $1 = \Sigma e_i$ in $(M,M)^A$ and decompositions $M = \bigoplus_i M_i$, characterized by the fact that e_i is the projection of M onto M_i with kernel $\bigoplus_{j\neq i} M_j$. Also $(M,M)^A$ is artinian, of course.

Theorem 5.2. Same notation as above. Then

i) (Krull-Schmidt.) The indecomposable direct summands of M are uniquely determined up to isomorphism. In other words, if

(1) $$M \simeq \bigoplus_{i\in I} M'_i \simeq \bigoplus_{j\in J} M''_j$$

where the M_i' 's and M_j'' 's are indecomposable A-modules, there exists a bijection $\phi : I \to J$ such that $M_i' \simeq M''_{\phi(i)}$ for all i.

ii) Let $M = M_1 \oplus M_2 = N_1 \oplus N_2$, and let $1 = e+1-e = f+1-f$ be the corresponding idempotent decompositions in $(M,M)^A$. Then $M_1 \approx N_1$ if and only if there exists a unit $u \in (M,M)^A$ such that $e = u^{-1}fu$.

Remark: Krull-Schmidt in fact holds even if A is only (right) artinian, but then the proof is no longer just an application of Theorem 3.12 (see for instance Curtis and Reiner (1981)), as we then lack the information that $(M,M)^A$ is artinian.

Proof: The decomposition in (1) corresponds to primitive idempotent decompositions in $(M,M)^A$. However, as we saw in the proof of Theorem 3.12 any such two decompositions are conjugate via a unit in $(M,M)^A$, from which i) follows.

ii) By Theorem 1.4, it suffices to prove that this holds for direct summands of E_E, where $E = (M,M)^A$. But again this follows from Theorem 3.12.

As an application of Theorem 3.14, we get

Lemma 5.3. Same notation as above. Then M is indecomposable if and only if $(M,M)^A/J((M,M)^A)$ is a division algebra over F.

Remark. A more general result, which again is beyond the scope of this book, states that if A is a ring and M is an A-module with a composition series, then M is indecomposable if and only if $(M,M)^A$ is local (see Curtis and Reiner (1981)).

Lemma 5.4. Let $e \in A$ be an idempotent and let M be any A-module. Then

$$(eA,M)^A \simeq Me \qquad (2)$$

as F-spaces.

Proof: We define $T : (eA,M)^A \to M$ by $T(\phi) = \phi(e)$. Then T is an F-linear map. Moreover, $\phi(e) = 0$ forces $\phi(e)a = \phi(ea) = 0$ for all $a \in A$, thus forcing $\phi = 0$, i.e., T is injective. Furthermore, $\phi(e) = \phi(e)e \in Me$, so T maps into Me. Conversely, if $x \in M$, we define $\phi_x \in (eA,M)^A$ by $\phi_x(a) = xa$. Then $T(\phi_x) = \phi_x(e) = xe$. Thus T is an isomorphism.

Corollary 5.5. Let $e \in A$ be an idempotent. Then

$$(eA,eA)^A \simeq eAe \tag{3}$$

and $T : (eA,eA)^A \to eAe$ defined by $T(\phi) = \phi(e)$ is an F-algebra isomorphism.

Lemma 5.6. Let $e \in A$ be an idempotent. Then

$$J(eAe) = eJ(A)e = J(A) \cap eAe. \tag{4}$$

Proof: The last equality is obvious. Moreover, $eJ(A)e$ is a nilpotent ideal in eAe, hence contained in $J(eAe)$. Finally,

$$(AJ(eAe)A)^r = (AeJ(eAe)eA)^r = AJ(eAe)^r A. \tag{5}$$

Thus $J(eAe)$ generates a nilpotent ideal in A, which shows that $J(eAe) \subseteq J(A)$.

This enables us to improve Lemma 5.1 for modules of the form eA, e an idempotent.

Corollary 5.7. Let $e \in A$ be an idempotent. Then

$$J((eA,eA)^A)^r \subseteq \{\phi \in (eA,eA)^A \mid \phi(eA) \subseteq eA \cap J(A)^r\} \tag{6}$$

for all r, and equality holds for $r = 1$.

Proof: By Lemma 5.6,

$$J(eAe)^r = (J(A) \cap eAe)^r \subseteq J(A)^r \cap eAe. \tag{7}$$

Moreover, $\phi \in J((eA,eA)^A)^r$ if and only if $\phi(e)$ $J(eAe)^r$ by Corollary 5.5, from which (6) follows. The equality for $r = 1$ is then obtained from Lemma 5.1.

<u>Lemma 5.8</u>. Assume F is a splitting field of $A/J(A)$, and let E_i be a simple A-module. Let M be an arbitrary A-module, and denote the multiplicity of E_i as a composition factor of M by a_i. Then

(8) $$a_i = \dim_F((P_i,M)^A)$$

where P_i is the p.i.m. corresponding to E_i.

<u>Proof</u>: Recall that by Theorem 3.14i) and Schur's lemma,

(9) $$(P_i,E_i)^A \simeq F$$

as an F-space. Let

(10) $$M = M_1 \supset M_2 \supset \dots \supset M_n = 0$$

be a filtration of M with M_j/M_{j+1} simple for all j. Choose j_1 maximal so that no composition factor of M/M_{j_1} is isomorphic to E_i. It immediately follows that

(11) $$W := (P_i,M)^A = (P_i,M^{\cdot}_{j_1})^A$$

as any factor module of P_i has E_i as a composition factor. Hence induction on $\dim_F M$ allows us to assume that $j_1 = 1$, and moreover that the dimension over F of $W_1 := (P_i,M_2)^A$ is a_i-1. Furthermore, as P_i is projective, there exists $\phi_1 \in W$ with $\phi_1 \in W_1$, or in other words

(12) $$(\phi_1(P_i)+M_2)/M_2 = M_1/M_2 \simeq E_i.$$

Now let $\phi \in W$ be arbitrary. Then (9) and (12) imply the existence of $\lambda \in F$ such that

$$([\phi-\lambda\phi_1](P_i) + M_2)/M_2 = 0. \tag{13}$$

Hence $\phi-\lambda\phi_1 \in W_1$, and we are done.

Corollary 5.9. Assume F is a splitting field of $A/J(A)$. Then the Cartan invariants c_{ij} of A satisfy

$$c_{ij} = \dim_F((P_j,P_i)^A) \tag{14}$$

with the notation of Section 4.

Remark. The proof above shows that in general, if F is arbitrary, then

$$c_{ij} \leq \dim_F((P_j,P_i)^A) \tag{15}$$

and equality holds if and only if $(E_i,E_i)^A \simeq F$, i.e., if and only if F is a spliting field of the Wedderburn component of $A/J(A)$ corresponding to E_i.

6. Duality.

We are now ready to take advantage of the fact that a group algebra over a finite group not only is a finite dimensional algebra but has a basis which forms a group! The following simple and yet extremely important definition takes advantage of that fact. It immediately leads us to a strong property of projective modules of a group algebra.

Definition 6.1. Let R be a commutative ring and G a finite group. By an $R[G]$-module M, we will always mean a module which considered as an R-module is free and finitely generated.

By the dual or contragredient, M^* of M, we understand

$$(M,R)^R \tag{1}$$

with the following action by G: for all $\phi \in (M,R)^R$ and all $g \in G$, we define

(2) $$(\phi g)(x) = \phi(xg^{-1})$$

for all $x \in M$. (The reader is urged to check that this makes $(M,R)^R$ into an $R[G]$-module of the same rank over R as M.) Finally, M is called self-dual if $M \simeq M^*$ as $R[G]$-modules.

Example: Let $R = \mathbb{C}$, and let χ be the character afforded by M. Then the character of M^* is $\bar{\chi}$.

We have the following evident properties of dual modules.

Lemma 6.2. Same notation as above. Then

i) $(M^*)^* \simeq M$.

ii) $(M \oplus N)^* \simeq M^* \oplus N^*$.

iii) M is indecomposable if and only if M^* is indecomposable.

iv) $M \simeq N$ if and only if $M^* \simeq N^*$.

Proof: Exercise.

Recall that for a group G, G and G^{op} are isomorphic, and $g \to g^{-1}$ is an isomorphism. For the same reason, we have

Theorem 6.3. Same notation as above. For any $g \in G$, define $\phi_g : R[G] \to R$ by

(3) $$\phi_g(\Sigma\alpha_g g) = \alpha_g.$$

Then $(R[G])^* = \underset{g}{\oplus} R\phi_g$, and $(\phi_g)h = \phi_{gh}$. Moreover

(4) $$\Sigma\alpha_g g \to \Sigma\alpha_g\phi_g$$

is an $R[G]$-isomorphism between $R[G]$ and $R[G]^*$. In other words, $R[G]$ is self-dual.

Proof. Only (4) has to be checked. But

(5) $$(\Sigma\alpha_g g)h = \Sigma\alpha_{gh^{-1}} g \to \Sigma\alpha_{gh^{-1}}\phi_g$$

while

(6) $$(\Sigma\alpha_g\phi_g)h = \Sigma\alpha_g\phi_{gh} = \Sigma\alpha_{gh^{-1}}\phi_g .$$

In fact, a more general result holds. Recall that if $H \leq G$ and N is an $R[H]$-module, the <u>induced module</u> $N^{\uparrow G}$, which is an $R[G]$-module, is defined as $N \otimes_{R[H]} R[G]$, where $R[G]$ is considered as a left $R[H]$-module. Alternatively, $N^{\uparrow G}$ is the R-module $\bigoplus_{g_i \in H\backslash G} N \otimes g_i$, where $H\backslash G$ denotes an arbitrary right transversal of H in G, with the following $R[G]$-action. For $g \in G$ and i arbitrary, let $g_i g = h_j g_j$. Then $(x \otimes g_i)g := xh_j \otimes g_j$ for all $x \in N$.

The following generalization of the fact that a group ring is self-dual is straightforward to check, but very important. It is actually a special case of a more general fact (see Chapter II, Lemma 1.2). We therefore omit the proof.

<u>Theorem 6.4</u>. Let $H \leq G$ and let N be an $R[H]$-module. Then

(7) $$(N^*)^{\uparrow G} \simeq (N^{\uparrow G})^* .$$

We now replace our arbitrary ring R by a field F. Consider any exact sequence of $F[G]$-modules

(10) $$0 \to X \to Y \to Z \to 0.$$

As we are dealing with vector spaces, the definition of a dual module implies that (10) induces an exact sequence

(11) $$0 \leftarrow X^* \leftarrow Y^* \leftarrow Z^* \leftarrow 0.$$

In particular, Theorems 2.2v) and 2.4iii) together with Lemma 6.2i) yield i) below.

<u>Lemma 6.5</u>. Let M be an $F[G]$-module. Then

i) M is projective if and only if M^* is injective.

ii) There is a one-to-one reversing correspondence between submodules of M and factor modules of M^*, and vica versa, induced by duality.

Using this and Theorem 6.3, we now obtain our first main goal of this section.

Theorem 6.6. An F[G]-module is projective if and only if it is injective.

Proof: Let $F[G] \simeq \oplus P_i$, where P_i is a p.i.m. for all i. As Hom is additive, Theorem 6.3 implies that $F[G] \simeq \oplus P_i^*$ as well. Thus P_i^* is projective for all i, and we are done by Lemma 6.5i).

Corollary 6.7. An F[G]-module is projective if and only if its dual is projective. In particular, a p.i.m. of $F[G]$ has a unique maximal submodule and a unique minimal submodule.

From this result, a very natural question arises: Is there any connection between the unique simple factor module and the unique simple submodule of a p.i.m. of $F[G]$? And indeed there is, as we proceed to show.

Theorem 6.8. The unique simple factor module of a p.i.m. of $F[G]$ is isomorphic to the unique simple submodule of that p.i.m.

The proof depends on the following observations:

Definition 6.9. By the augmentation map of $F[G]$, we understand the linear map $\lambda : F[G] \to F$ given by

(12) $$\lambda(\Sigma \alpha_g g) = \alpha_1.$$

This map has the following important properties.

Lemma 6.10. i) Ker λ does not contain any left or right ideal of $F[G]$.

ii) For all $a, b \in F[G]$, $\lambda(ab) = \lambda(ba)$.

Proof: i) Assume $\lambda(aF[G]) = 0$. In particular, $\lambda(ag^{-1}) = 0$ for all $g \in G$. But $\lambda(ag^{-1}) = \alpha_g$, where $a = \Sigma\alpha_g g$, which forces $a = 0$.

ii) Let $a = \Sigma\alpha_g g$ and $b = \Sigma\beta_h h$. Then

$$\lambda(ab) = \sum_{gh=1} \alpha_g\beta_h = \sum_{hg=1} \beta_h\alpha_g = \lambda(ba). \tag{13}$$

Proof of Theorem 6.8. We may now prove the claimed property of a p.i.m. P of $F[G]$. Let e be a primitive idempotent in $F[G]$ with $P \simeq eF[G]$, and denote the simple submodule of $eF[G]$ by E_1, the simple factor module by E_2. In particular, $eE_1 = E_1$. It suffices to prove that $E_1e \neq 0$. Indeed, as we saw in Lemma 5.4, this forces $E_1 = E_1eF[G]$ to be a homomorphic image of $eF[G]$, and now uniqueness of E_1 and E_2 forces $E_1 \simeq E_2$. Finally, that in fact $E_1e \neq 0$ is an easy consequence of Lemma 6.10: Choose by i) $a \in E_1$ with $\lambda(a) \neq 0$. As $a = ea$, ii) states that $\lambda(ae) = \lambda(a)$. Hence $\lambda(ae) \neq 0$ and thus a priori $ae \neq 0$.

Remark: The only fact needed to prove Lemma 6.5 was that duality preserves exact sequences. If we therefore return to $R[G]$ for an arbitrary commutative ring R, the dual (11) of an exact sequence (10) will indeed be an exact sequence if the image of X in Y is a direct summand of Y as an R-module. Thus projective $R[G]$-modules are injective as well in the category of free R-modules. This will be further explained in Section 14.

7. Symmetry.

Finite algebras with the property that a module is projective if and only if it is injective are called quasi-frobenius. We do not intend to discuss this further here, but refer the reader to, for instance, Curtis and Reiner (1966) for positively all aspects of this definition. We mention one interesting connection to group algebras though, namely a more recent result of Green (1978b), based on a theorem due to Sawada (1977), which states the following: Let G be a finite group with a split (B,N)-pair, and let $U = B \cap N$, which is a Sylow p-subgroup of G. Then $(I^{\uparrow G}, I^{\uparrow G})^{F[G]}$, where F is a field of characteristic p and I is the trivial $F[U]$-module, is a quasi-frobenius algebra.

Here we want to consider a slightly more restricted class of algebras, namely symmetric algebras. The motivation for this is first of all, as we shall see, that the endomorphism ring of any direct sum of p.i.m.'s of a group algebra is a symmetric algebra.

Readers with a primary background in ring theory may have been caught by slight surprise, when we defined the dual of a right module as a right module. The standard way of defining the dual of a right module of a finite dimensional algebra is the following.

Definition 7.1. Let A be a finite dimensional algebra over the field F, and let M be a right A-module. By the (F-) dual ${}^*M$ of M, we understand $(M,F)^F$ with the following left action of A: For all $\phi \in (M,F)^F$ and all $a \in A$, we define

(1) $$(a\phi)(m) = \phi(ma)$$

for all $m \in M$.

We may change this to a right module exactly in the case when $A \simeq A^{op}$, the opposite ring of A, in the following obvious way: If $op : A \to A^{op}$ denotes the isomorphism, we define $(\phi a)(m) = \phi(m\ op(a))$, which is exactly what happened in the previous section. Thus, the fact that we, in certain cases, may introduce the dual of a right module as a right module rather than the general left is completely irrelevant as far as the properties listed in Lemmas 6.2 and 6.5 are concerned. For the sake of completeness, we state

Proposition 7.2. Same notation as above. Consider any exact sequence of A-modules

(2) $$0 \to X \to Y \to Z \to 0.$$

Then the induced sequence

(3) $$0 \leftarrow {}^*X \leftarrow {}^*Y \leftarrow {}^*Z \leftarrow 0$$

is exact as well. Moreover, if M and N are arbitrary A-modules, the following holds:

i) $({}^*M)^* \simeq M$.

ii) ${}^*(M \oplus N) \simeq {}^*M \oplus {}^*N$.

iii) $M \simeq N$ if and only if ${}^*M \simeq {}^*N$.

iv) M is projective if and only if ${}^*M$ is injective.

v) There is a one-to-one reversing correspondence between submodules of M and factor modules of ${}^*M$ as described by (2) and (3), and vice versa.

Before we get carried away with general observations, let us introduce

Definition 7.3. Same notation as above. Then A is called a frobenius algebra if there exists a linear map $\lambda : A \to F$ with the property stated in Lemma 6.10i),

i) $\operatorname{Ker} \lambda$ does not contain any left or right ideals of A.

If furthermore λ has the property stated in Lemma 6.10ii),

ii) For all $a, b \in A$, $\lambda(ab) = \lambda(ba)$.

Then A is called symmetric.

Thus Lemma 6.10 may be reformulated as

Lemma 7.4. A group algebra over a finite group is a symmetric algebra.

With this definition, the following does not come as a big surprise.

Proposition 7.5. Let A be a frobenius algebra. Then

i) $A_A \simeq ({}_AA)^*$ as A-modules.

ii) An indecomposable A-module is projective if and only if its dual is projective.

If furthermore A is symmetric, we have

iii) Let P be a p.i.m. of A. Then the unique simple factor module of P is isomorphic to the unique simple submodule.

Proof: Once i) is established, ii) and iii) follow exactly as in the case of a group algebra. Thus we only have to prove i), which provides us with a different proof of Theorem 6.6. Or does it?

To prove i), we define for every x A an F-linear map $\lambda_x : {}_AA \to F$ by $\lambda_x(y) = \lambda(xy)$, where λ is defined as in Definition 7.3. Then $\lambda_x \in ({}_AA)^*$ and we define $\phi : A_A \to ({}_AA)^*$ by $\phi(x) = \lambda_x$. This is obviously an F-linear map, and Definition 7.3 i) ensures that ϕ is injective and hence bijective. Moreover, if $a \in A$,

(4) $$\lambda_{xa}(y) = \lambda(xay) = \lambda_x(ay) = [\lambda_x a](y)$$

which shows that ϕ is A-linear.

Finally, the main result of this section:

Theorem 7.6. Let A be a symmetric algebra and let $e \in A$ be an idempotent. Then $(eA,eA)^A$ $(\simeq eAe)$ is symmetric.

Proof: Choose λ as in Definition 7.3, and consider the restriction of λ to eAe. Clearly, ii) is still satisfied. To establish i), let $x \in eAe$ and assume $\lambda(xeAe) = 0$. As $x \in eAe$ however, $x = exe$ and thus

(5) $$0 = \lambda(xeAe) = \lambda(exeA) = \lambda(xA)$$

by ii). Hence $x = 0$, and ii) holds for the restriction of λ to eAe.

8. Loewy series and socle series.

Again the following basic definitions go back to Artin, Nesbitt and Thrall (1944). Note though, that here the Loewy and socle series are called the upper and lower Loewy series.

Definition 8.1. Let A be an artinian ring and denote its radical by J. Let M be an arbitrary (but always finitely generated) A-module, and choose m minimal such that $MJ^m = 0$. Recall that M/MJ is the maximal semisimple factor module of M. By the Loewy series, we mean

$$\begin{array}{l} M/MJ \\ MJ/MJ^2 \\ \cdot \\ \cdot \\ \cdot \\ MJ^{m-2}/MJ^{m-1} \\ MJ^{m-1} \end{array} \tag{1}$$

If $MJ^i/MJ^{i+1} \simeq X_{i+1,1} \oplus \dots \oplus X_{i+1,r_i}$, where $X_{i+1,j}$ is simple for all j, we usually write

$$\begin{array}{l} X_{11} \cdots\cdots\cdots X_{1r_1} \\ X_{21} \cdots\cdots\cdots\cdots\cdots X_{2r_2} \\ \cdot \\ \cdot \\ \cdot \\ X_{m1} \cdots X_{mr_m} \end{array} \tag{2}$$

Likewise, we define the <u>socle</u> <u>series</u> in the following way. Let $S_1(M)$ denote the maximal semisimple submodule of M and define $S_i(M)$ successively by

$$S_i(M)/S_{i-1}(M) = S_1(M/S_{i-1}(M)). \tag{3}$$

By the socle series we mean

$$\begin{array}{l} S_n(M)/S_{n-1}(M) \\ \cdot \\ \cdot \\ \cdot \\ S_2(M)/S_1(M) \\ S_1(M) \end{array} \tag{4}$$

where n is chosen minimal with $S_n(M) = M$. We usually write (4) in a way similar to (2).

$S_1(M)$ is also called the <u>socle</u> of M and often denoted by $Soc(M)$. Similarly, we define the <u>head</u> of M as $Hd(M) := M/MJ$. Finally, we recall that $MJ =: J(M)$ is called the radical of M.

Lemma 8.2. Let A and M be as above. Then

i) $n = m$. This common number is called the Loewy length of M and is denoted by $j(M)$.

ii) $MJ^r \subseteq S_{m-r}(M)$ for all r if we set $S_0(M) = 0$.

Proof: Easy exercise.

Lemma 8.3. Let A be an artinian ring of Loewy length n. Then $S_i(A)$ is the annihilator in A of $J^i(A)$. In other words,

$$(5) \qquad S_i(A) = \{a \in A \mid aJ^i(A) = 0\}.$$

In particular, $S_i(A)$ is a 2-sided ideal of A.

Proof: By definition of $S_i(A)$, $S_i(A)J(A) = S_{i-1}(A)$ in view of Corollary 3.7, from which (5) follows. In particular, $S_i(A)$ is a 2-sided ideal as $J(A)$ is.

Lemma 8.4. Let F be a field and A a finite dimensional algebra over F. Let M be an A-module. Then

i) The socle series of M is the dual of the Loewy series of the F-dual of M, ${}^*M$ (or M^* if A is a group algebra).

ii) M and ${}^*M$ have the same Loewy length.

Proof: i) follows from the fact that taking duals reverses exact sequences, and ii) follows from i) as $({}^*M)^* \simeq M$ as A-modules.

Lemma 8.5. Same notation as in Lemma 8.2.

i) Let $N \subseteq M$. Then the socle series of N is obtained by intersecting N with the socle series of M.

ii) (Stripping a factor of the socle series) M contains a simple module S in its i'th socle if and only if M has a submodule of Loewy length i with simple head S.

Proof: i) By definition of socle series, $S_i(M) \cap N \subseteq S_i(N)$. On the other hand, $S_1(N)$ is semisimple, hence contained in $S_1(M)$.

Assume therefore by induction that $S_{i-1}(N) \subseteq S_{i-1}(M)$. Then

(6) $$(S_i(N) + S_{i-1}(M))/S_{i-1}(M)$$

is semisimple, i.e., $S_i(N) + S_{i-1}(M) \subseteq S_i(M)$, and we are done.

ii) follows from i) and the universal property of P_S, the p.i.m. with head S.

Lemma 8.6. Assume A is a symmetric algebra. Then the following dual statements of those of Lemma 8.4 hold:

i) Let $N \subseteq M$. Then the Loewy series of M/N is the homomorphic image of that of M by the canonical map $M \to M/N$.

ii) (Stripping a factor of the Loewy series) M contains a simple module S in its i'th Loewy layer if and only if M has a factor module of Loewy length i with simple socle S.

Remark. Let X be an A-module, and assume the Loewy resp. socle series of X has the form

(7) $$\begin{matrix} E_1 \\ E_2E_3 \\ E_4 \\ E_5 \end{matrix} \qquad \begin{matrix} E_1 \\ E_2 \\ E_3E_4 \\ E_5 \end{matrix}$$

where E_i is simple for all i. As $XJ = S_3(X)$ and $XJ^3 = S_1(X)$, we deduce that $XJ/S_1(X)$ has Loewy resp. socle series.

(8) $$\begin{matrix} E_2E_3 \\ E_4 \end{matrix}, \qquad \begin{matrix} E_2 \\ E_3E_4 \end{matrix}$$

which shows that $XJ/S_1(X) \simeq \begin{matrix} E_2 \\ E_4 \end{matrix} \oplus E_3$, where we identify the first component by its Loewy (and socle) series.

If A is a finite dimensional algebra over F and we denote a matrix-representation corresponding to an A-module M by $\underline{\underline{M}}$, this shows that $\underline{\underline{X}}$ may be chosen to be of the form

(9) $$\begin{Bmatrix} \underline{\underline{E}}_5 & 0 & & & 0 \\ * & \underline{\underline{E}}_4 & & & \\ * & 0 & \underline{\underline{E}}_3 & & \\ * & * & 0 & \underline{\underline{E}}_2 & 0 \\ * & * & * & * & \underline{\underline{E}}_1 \end{Bmatrix}$$

Definition 8.7. Same notation as above. Then M is called uniserial, if MJ^{i-1}/MJ^{i} is simple for all $j \leq j(M)$.

Example. Let $P = \langle x \rangle$ be a cyclic p-group of order p^a, and let M be any indecomposable $F[P]$-module. Using the Jordan Canonical form of x, we see that

i) $\dim_F M \leq p^a$.

ii) For each integer $m \in \{1,2,\ldots,p^a\}$, there exists, up to isomorphism, exactly one indecomposable $F[P]$-module of dimension m.

iii) M is uniserial.

9. The p.i.m.'s.

Let A be a finite dimensional algebra over the field F. We have discussed the one-to-one correspondence between the isomorphism classes of the simple A-modules with representatives $E_1,\ldots,E_r$ and the isomorphism classes of the p.i.m.'s of A with representatives $P_1,\ldots,P_r$, characterized by the fact that $P_i/P_iJ(A) \simeq E_i$. Also, the multiplicity of P_i as a direct summand of A equals that of E_i as a direct summand of $A/J(A)$, which in the case of F being a splitting field of $A/J(A)$ equals $\dim_F E_i$.

The Cartan invariants c_{ij} of A have been defined as the multiplicity of E_j as a composition factor of P_i, and we have shown that if F is a splitting field of $A/J(A)$, then

(1) $$c_{ij} = \dim_F((P_j,P_i)^A).$$

If furthermore A is symmetric, we have proved that $c_{ii} \geq 2$ unless $P_i = E_i$. As moreover P_i is injective, we have the following characterization of this case.

Proposition 9.1. Let A be a symmetric algebra and use the notation above. Let $\mathbb{B}$ be a block of A in which E_i lies, i.e., $\mathbb{B}$ is the unique block for which $E_i\mathbb{B} = E_i$. Then the following are equivalent:

i) $c_{ii} = 1$.

ii) E_i is projective (i.e., $E_i \simeq P_i$).

iii) $\mathbb{B}_{\mathbb{B}} \simeq E_i^{(n_i)}$ for some n_i.

We now finally want to assume that A is in fact a group algebra and remind the reader of the following as our starting point.

Proposition 9.2. let G be a finite group, F a field of characteristic p and Q a normal p-group of G. Then $F[Q]$ acts trivially on any simple $F[G]$-module. In particular, the trivial module is the only simple module of $F[G]$, if G is a p-group.

This property in fact characterizes p-groups. Namely,

Proposition 9.3. Let G be a finite group and F a field of characteristic p. Then the following are equivalent

i) G is a p-group.

ii) $F[G]_{F[G]}$ is indecomposable.

iii) $\dim_F(F[G]/J(F[G])) = 1$.

Proof: The equivalence of ii) and iii) is a consequence of the one-to-one correspondence between simple modules and p.i.m.'s. Also, i) implies iii) by Proposition 9.2. Conversely, assume the trivial module I is the only simple $F[G]$-module. Then the composition factors of any module are isomorphic to I. Let $x \in G$ be an element of order prime to p. As $F[G]$ is a free $F[\langle x\rangle]$-module, it is isomorphic to a direct sum of modules isomorphic to $F[\langle x\rangle]_{F[\langle x\rangle]}$ as an $F[\langle x\rangle]$-module. But as $F[\langle x\rangle]$ is semisimple, x acts on $F[\langle x\rangle]$ as the identity then, forcing $x = 1$. Thus G is a p-group.

Note that implicit we use the following obvious but extremely important fact.

Proposition 9.4. Same notation as above. Let P be a projective F[G]-module, and let $H \leq G$. Then P is a projective F[H]-module.

Proof: F[G] is a free F[H]-module.

Corollary 9.5. Let G be an arbitrary group, let $Q \in Syl_p(G)$ and let F be a field of characteristic p. Let P be any projective F[G]-module. Then $|Q|$ divides $\dim_F(P)$.

Proof: By Proposition 9.4 and 9.5.

Corollary 9.6. Same notation as in Corollary 9.5. Assume furthermore that Q is normal in G. Then

i) $J(F[G]) = F[G]J(F[Q])$.

ii) Let M be an F[G]-module. Then the Loewy resp. socle series of the restriction of M to F[S] is the restriction of the Loewy resp. socle series of M.

Proof: By duality, it suffices to prove ii) for the Loewy series. But this is a direct consequence of i) using the basic properties of the radical.

To prove i) denote the radical of $A = F[G]$ by J. As $Q \trianglelefteq G$, Q is in the kernel of any simple A-module by Proposition 9.2. Hence A/J is a semisimple F[Q]-module (indeed a trivial F[Q]-module) and thus $AJ(F[Q]) \subseteq J$. A fast way of proving the other inclusion is the following: By a theorem of Schur, Q has a complement K, and by Proposition 9.3,

$$A/AJ(F[Q]) \simeq F[K] \tag{2}$$

which is a semisimple algebra and hence a semisimple A-module. Thus $J \subseteq AJ(F[Q])$ by the properties of the radical.

With these results, we have exhausted the more obvious general remarks.

We proceed to investigate the Cartan matrix of a symmetric algebra A over a field F. Denote the radical of A by J, and assume F is a splitting field of A/J. As before $P_1,\dots,P_r$ will be a complete set of representatives of the p.i.m.'s of A, and $E_i = P_i/P_iJ$.

<u>Lemma 9.7</u>. Choose $s \in \mathbb{N}$ arbitrary, and let $\{\beta_1,\dots,\beta_{m_s}\}$ be a basis of a complement to $(P_i/P_iJ^{s-1},P_j)^A$ in $(P_i/P_iJ^s,P_j)$. Set

(3) $$S_i/P_iJ^{s-1} = \bigcap_{r=1}^{m_s} \operatorname{Ker} \beta_r, \quad S_j = \sum_{r=1}^{m_s} \operatorname{Im} \beta_r.$$

Then

(4) $$m_s = \dim_F((E_j,P_i/S_i)^A) = \dim_F((S_j,E_i)^A).$$

<u>Proof</u>: The first equality follows from the fact that by assumption

(5) $$m_s = \dim_F((E_j,P_iJ^{s-1}/P_iJ^s)^A)$$

as A is symmetric. Moreover, for any $\beta = \Sigma\lambda_r\beta_r$, $\beta(P_i)$ is of Loewy length s by Lemma 8.5ii). In particular, if η is the canonical homomorphism $S_j \to S_j/S_jJ$, then $\eta\beta_1,\dots,\eta\beta_{m_s}$ are linearly independent, i.e., $S_j/S_jJ \simeq (E_i)^{(m_s)}$, which is equivalent to the second equality.

As a corollary, we obtain

<u>Theorem 9.8</u>. Let A be a symmetric algebra over the field F, and assume F is a splitting field of $A/J(A)$. Then the Cartan matrix of A is symmetric.

<u>Proof</u>: By considering the Loewy series of P_i we see that $c_{ij} = \sum_s m_s$ with the notation of Lemma 9.7. By considering the socle series of P_j we see that $c_{ji} = \sum_s m_s$ as well.

<u>Remark</u>. It is necessary to assume that F is a splitting field of $A/J(A)$. We will give a counter example to the general case of Theorem 9.8 in Section 18.

For the sake of simplicity, we now assume that A is in fact a group algebra. The advantage of this is that the dual of a right module again is a right.

Thus $A = F[G]$, where G is a finite group and F is assumed to be a splitting field of A/J, where $J = J(F[G])$. Denote the dual module of E_i by E_{i^*}. Then $P_i^* \simeq P_{i^*}$, as $F[G]$ is symmetric. Now let $\mathfrak{n} \neq 0$ be any power of J and set $\bar{A} = A/\mathfrak{n}$, $\bar{P}_i = P_i/P_i\mathfrak{n}$. Then $\bar{A}$ is a finite dimensional algebra but certainly not symmetric or even quasi-frobenius in general. However, as $E_i\mathfrak{n} = 0$ for all i, $\{E_1,\ldots,E_r\}$ is still a complete set of representatives of the isomorphism classes of simple $\bar{A}$-modules, and $\bar{P}_i$ is the p.i.m. of $\bar{A}$ corresponding to E_i. Denote the Cartan matrix of $\bar{A}$ by $\{\bar{c}_{ij}\}$. Now the following holds.

Theorem 9.9 (Landrock (1983)). The Cartan matrix of $\bar{A}$ is dual symmetric, i.e., $\bar{c}_{ij} = \bar{c}_{j^*i^*}$.

This will follow from the following result, which is interesting in itself.

Lemma 9.10. Let $s \in \mathbb{N}$ be arbitrary. Then the multiplicity of E_j as a composition factor of P_iJ^{s-1}/P_iJ^s equals that of E_{i^*} in $P_{j^*}J^{s-1}/P_{j^*}J^s$.

Proof: Using duality, it suffices to prove that the first number, a_1, is less than or equal to the second, a_2. By Lemma 9.7,

$$a_1 = \dim_F((E_j, P_i/P_iJ^s)^A) = \dim_F((S,E_i)^A) \tag{6}$$

where S is a submodule of P_j of the form $S = \sum_{r=1}^{m_s} V_r$ and each V_r is a homomorphic image of P_i in P_j. Moreover, and this is the vital part, any submodule of S not contained in SJ is of Loewy length s. Or, in other words, $SJ = S_{s-1}(S)$. But this implies that S^* is a quotient module of P_{j^*} with $S^*J^{s-1} \simeq (E_{i^*})^{(a_1)}$. Hence Lemma 8.5 asserts that $a_1 \leq a_2$, and we are done.

Proof of Theorem 9.9: Denote the Cartan matrix of A/J^s by $\{c_{ij}(s)\}$. It suffices to prove that

$$c_{ij}(s) - c_{ij}(s-1) = c_{j^*i^*}(s) - c_{j^*i^*}(s-1) \tag{7}$$

for all s, which is exactly the statement of Lemma 9.10.

Remark. Of course a similar result holds for the socle series of P_i and P_j. For another result, see Lemma II.1.5.

Lemma 9.10 has a number of applications to the Loewy structure of the p.i.m.'s of a group algebra, of which we only mention:

Corollary 9.11. Let $Q_1,\ldots,Q_t$ denote the p.i.m.'s of some block of $F[G]$, and assume the Loewy series of $Q_2,\ldots,Q_t$ are known. Then the Loewy series of Q_1^* is known as well except for the composition factors isomorphic to $(Q_1/JQ_1)^*$.

10. Ext.

Definition 10.1. Let A be an artinian ring and let M be an arbitrary A-module. Choose P_M projective with $Hd(P_M) \simeq Hd(M)$. Then P_M is called the projective cover of M.

Assume furthermore that A is a quasi-frobenius algebra. Choose I_M projective (= injective) so that $Soc(I_M) \simeq Soc(M)$. Then I_M is called the injective hull of M.

From the definition, we immediately obtain

Lemma 10.2. With the notation above,

i) There exists a surjective homomorphism $P_M \xrightarrow{\alpha} M$.

ii) There exists an injective homomorphism $M \xrightarrow{\beta} I_M$.

Proof: i) follows from Nakayama's lemma, and ii) from i) by duality.

<u>Lemma 10.3</u> (Schanuel's lemma). Same notation as above. Let P_1, P_2 be projective A-modules such that

(1) $$0 \to W_1 \to P_1 \overset{\phi_1}{\to} V \to 0$$

(2) $$0 \to W_2 \to P_2 \overset{\phi_2}{\to} V \to 0$$

are both exact. Then

(3) $$W_1 \oplus P_2 \simeq W_2 \oplus P_1.$$

<u>Proof</u> (W. Feit): Let U be the submodule of $P_1 \oplus P_2$ defined by

(4) $$U := \{(x_1, x_2) \mid \phi_1(x_1) = \phi_2(x_2)\}$$

and let ε_1 be the projection of $P_1 \oplus P_2$ onto P_1 with kernel P_2. Then $\varepsilon_1(U) = P_1$ and

(5) $$U \cap \operatorname{Ker} \varepsilon_1 = \{(0, x_2) \mid x_2 \in \operatorname{Ker} \phi_2\} \simeq W_2.$$

Now projectivity of P_1 asserts that $U \simeq W_2 \oplus P_1$. By the same argument, $U \simeq W_1 \oplus P_2$.

<u>Notation</u>. Let A be an artinian ring, M an A-module. We then define the module ΩM by

(6) $$0 \to \Omega M \to P_M \overset{\alpha}{\to} M \to 0.$$

If furthermore A is quasi-frobenius, we define the module $\mho M$ by

(7) $$0 \to M \overset{\beta}{\to} I_M \to \mho M \to 0.$$

These operators are called the <u>Heller operators</u>.

<u>Corollary 10.5</u>. Let A be an artinian ring, and let M and P be A-modules with P projective. Then

i) ΩM is uniquely determined up to isomorphism, independently of the choice of α.

ii) Assume $\varphi : P \to M$ is surjective. Then $P \simeq P_M \oplus P'$, where $\ker \varphi \simeq \Omega M \oplus P'$.

iii) Let N be a submodule of P and assume M is a direct summand of P/N. Then ΩM is a direct summand of N.

<u>Remark</u>. Of course, similar statements hold about $\mho M$ if A is quasi-frobenius.

<u>Proof</u>: i) follows directly from Schanuel's lemma.

ii) Here we get at least that $\Omega M \oplus P \simeq \ker \varphi \oplus P_M$. We now cheat and use Krull-Schmidt. Our justification is that 1) We will only apply it in the case where A is a finite dimensional algebra, 2) Krull-Schmidt actually does hold for artinian rings as pointed out earlier. An immediate consequence is that P_M is isomorphic to a direct summand of P and we are done.

iii) Follows from ii).

<u>Exercise</u>: Find a valid proof of ii) and iii) above.

Before we turn to the main idea of this section, we list some basic properties of the Heller operators.

<u>Lemma 10.6</u>. Let A be a quasi-frobenius algebra, and let M be a projective-free A-module. Then

i) Ω is additive on modules.

ii) $\mho(\Omega M) \simeq \Omega(\mho M) \simeq M$.

iii) ${}^*(\Omega M) \simeq \mho({}^*M)$ and ${}^*(\mho M) \simeq \Omega({}^*M)$.

iv) (Heller's lemma) $M \simeq \Omega(\Omega({}^*M)^*)$.

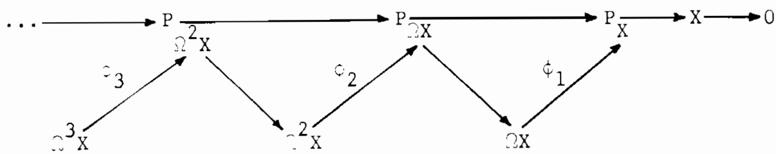
v) M is indecomposable if and only if ΩM is indecomposable.

<u>Proof</u>: i), ii), and iii) are direct consequences of the definition of ΩM and $\mho M$, and the fact that exact sequences are preserved under taking duals.

We are now ready to define the Ext-group.

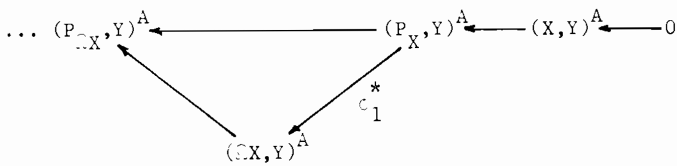
Definition 10.7. Let X and Y be A-modules, where A is an artinian ring. Consider the exact sequence

(8)



where $\Omega^n X := \Omega(\Omega^{n-1}X)$. This induces a sequence

(9)



We now define

(10) $$\mathrm{Ext}_A^n(X,Y) := (\Omega^n X,Y)^A/\phi_n^*((P_{\Omega^{n-1}X},Y)^A)$$

$$= \mathrm{Ext}_A^{n-1}(\Omega X,Y)$$

$$\vdots$$

$$= \mathrm{Ext}_A^1(\Omega^{n-1}X,Y)$$

Example 1: Let J denote the radical of A and assume E_1 and E_2 are simple A-modules. Let $\psi \in (P_{E_1},E_2)^A$. Then $P_{E_1}J \subseteq \mathrm{Ker}\ \psi$ as E_2 is simple. Consequently, if we set $E_1 = X$ and $E_2 = Y$ above, we obtain that $\psi \circ \phi_1 = 0$. Thus

(11) $$\mathrm{Ext}_A^1(E_1,E_2) \cong (P_{E_1}J,E_2)^A$$

whenever E_1 and E_2 are simple. This example indicates a connection between possible extensions of E_1 by E_2 and the size of $\mathrm{Ext}_A^1(E_1,E_2)$. If the reader is unfamiliar with this, he or she is referred to the appendix on Ext.

Next we explain how properties of Ext_A^1-groups of simple modules relate to basic properties of A.

Proposition 10.8. Let A be a symmetric algebra and let E_1 and E_2 be simple A-modules. Then E_1 and E_2 lie in the same block of A if and only if there exists a sequence of simple modules

(12) $$E_1 = T_1, T_2, \ldots, T_n = E_2$$

such that $\mathrm{Ext}_A^1(T_i, T_{i+1}) \neq 0$ for all i.

Proof: We first observe, in view of the example above that if M is any A-module and S_2 is any simple composition factor of MJ^r/MJ^{r+1} for some r, where $J = J(A)$, then there exists a simple composition factor S_1 of MJ^{r-1} such that $\mathrm{Ext}_A^1(S_1, S_2) \neq 0$, as $\mathrm{Hd}(MJ^{r-1}/MJ^{r+1}) = MJ^{r-1}/MJ^r$.

Now, if the assumption above holds, then P_i and P_{i+1} are in the same block by Definition 4.3, where P_i denotes the projective cover of T_i, and consequently E_1 and E_2 belong to the same block.

Conversely, if E_1 and E_2 belong to the same block, choose by definition p.i.m.'s

(13) $$P_{E_1} = P_1, P_2, \ldots, P_m = P_{E_2}$$

such that P_i and P_{i+1} have a simple composition factor S_i in common. Denote $\mathrm{Hd}(P_i) \simeq \mathrm{Soc}(P_i)$ by U_i. By the remark above, we may find sequences

(14) $$S_i = X_1, \ldots, X_{s_i} = U_i, \quad U_i = Y_1, \ldots, Y_{t_i} = S_{i+1}$$

for all i such that $\mathrm{Ext}_A^1(X_j, X_{j+1}) \neq 0$ for all j and $\mathrm{Ext}_A^1(Y_k, Y_{k+1}) \neq 0$ for all k, from which (13) then follows.

Corollary 10.9. Assume the socle series of any p.i.m. of a block $\mathbb{B}$ of a symmetric algebra equals the Loewy series. Then all p.i.m.'s of $\mathbb{B}$ have the same Loewy length.

<u>Proof</u>: Let Q_1 and Q_2 be arbitrary p.i.m.'s of $\mathbb{B}$. It suffices to prove that $j(Q_1) \leq j(Q_2)$. Let $S_i = Hd(Q_i)$ and choose a sequence of simple modules as in (12). Again we denote the projective cover of T_i by P_i. Thus it suffices to prove that $j(P_i) \leq j(P_{i+1})$ for all i. However, by (11), T_{i+1} is a simple composition factor of P_iJ/P_iJ^2, where J is the radical of $\mathbb{B}$, and hence by assumption of $S_{m-1}(P_i)/S_{m-2}(P_i)$, where m is the Loewy length of P_i. Thus P_i has a submodule of Loewy length $m - 1$ with simple head T_{i+1}, which shows that the Loewy length of P_{i+1} is at least m.

For this and related results, the reader is referred to Landrock (1980). Also we mention without proof that our assumption on the p.i.m.'s in Corollary 10.9 is satisfied for the group algebra of an arbitrary p-group. This was proved in Jennings (1941). However, in this case Corollary 10.9 is vacuous, as the group algebra is indecomposable as a module. Also it follows then from Corollary 9.6 that the p.i.m.'s of any group with a normal Sylow p-subgroup have identical Loewy and socle series. Also, the p.i.m.'s of $SL(2,p^n)$ in characteristic p enjoy this remarkable property, as proved in Andersen, Jørgensen and Landrock (1983), which definitely not is shared by most other groups of Lie type.

Finally, we prove two important results on relations between G/G' and $\mathrm{Ext}^1_{F[G]}(I,I)$, where G is an arbitrary finite group and I is the trivial $F[G]$-module.

Consider the exact sequence

(15) $$0 \to \Omega\mathbf{Z} \to \mathbf{Z}[G] \overset{\varepsilon}{\to} \mathbb{Z} \to 0$$

where the <u>augmentation ideal</u> $\Omega\mathbf{Z}$ is defined by

(16) $$\Omega\mathbf{Z} = \mathrm{span}_{\mathbf{Z}}\{g-1 \mid g \in G\}$$

and ε is the canonical map. With this notation, we have

<u>Lemma 10.10</u>. The abelian group $\Omega\mathbf{Z}/(\Omega\mathbf{Z})^2$ is isomorphic to G/G'.

Proof: (See Hilton and Stambach (1971).) As $\Omega\mathbf{Z}$ is free over $\{g-1 \mid g \in G\}$,

(17) $$\alpha_1 : g-1 \to gG'$$

extends to a group homomorphism $\alpha_1 : \Omega\mathbf{Z} \to G/G'$. (Observe that $\alpha_1(1-g) = g^{-1}G'$.) As

(18) $$(x-1)(y-1) = (xy-1) - (x-1) - (y-1)$$

we see that $(\Omega\mathbf{Z})^2 \subseteq \text{Ker } \alpha_1$. Conversely, if we define $\alpha_2 : G \to \Omega\mathbf{Z}/(\Omega\mathbf{Z})^2$ by

(19) $$\alpha_2 : g \to g - 1 + \Omega\mathbf{Z}$$

then (18) shows that α_2 is a group homomorphism, and as $\Omega\mathbf{Z}/(\Omega\mathbf{Z})^2$ is abelian, $G' \leq \text{Ker } \alpha_2$. On the other hand, as obviously α_1 and α_2 are the inverses of each other modulo their kernels, the statement holds.

Corollary 10.11. Let P be a p-group, and let $\Phi(P)$ denote its Frattini subgroup. Let F be a field of characteristic p. Then

(20) $$\dim_F(\text{Ext}^1_{F[P]}(I,I)) = \text{rank}(P/\Phi(P))$$

where I is the trivial $F[P]$-module.

Proof: Denote by $GF(p)$ the field with p elements. Then

(21) $$\Omega\mathbf{Z} \otimes_{GF(p)} F = \Omega I = J(F[P])$$

and the statement follows since for an abelian p-group A we have that $A \otimes_{\mathbf{Z}} GF(p) \simeq A/\Phi(A)$.

Corollary 10.12. Same notation as above. Let $x_1,\ldots,x_r$ be a minimal set of generators of P. Then

(22) $$J := J(F[P]) = (1-x_1,\ldots,1-x_r)$$

and

(23) $$\{1-x_1+J^2,\ 1-x_2+J^2, \dots, 1-x_r+J^2\}$$

form a basis of J/J^2.

<u>Corollary 10.13</u>. Let G be any group, F a field of characteristic p, and let I denote the trivial $F[G]$-module. Then

(24) $$\mathrm{Ext}^1_{F[G]}(I,I) \simeq \mathrm{Hom}(G,\mathbb{Z}_p)\otimes_{GF(p)}F$$

where $\mathbb{Z}_p$ is the cyclic group of order p. In particular, $\mathrm{Ext}^1_{F[G]}(I,I) \neq 0$ if and only if G has a normal subgroup of index p.

<u>Proof</u>: Denote the trivial $GF(p)[G]$-module by I_p. As

(25) $$\mathrm{Ext}^1_{F[G]}(I,I) \simeq \mathrm{Ext}^1_{GF(p)}(I_p,I_p)\otimes_{GF(p)}F,$$

we may as well assume that $F = GF(p)$. By (11),

(26) $$\mathrm{Ext}^1_{GF(p)[G]}(I,I) \simeq (\Omega\mathbf{Z} \otimes_{\mathbf{Z}} GF(p), I_p)^{GF(p)[G]}.$$

Let $\varphi : \Omega\mathbf{Z} \otimes_{\mathbf{Z}} GF(p) \to I_p$ be a $GF(p)[G]$-homomorphism. Then

(27) $$\varphi((x-1)y) = \varphi(x-1)y = \varphi(x-1)$$

for all $x,y \in G$. Thus $\varphi((x-1)(y-1)) = 0$ and hence

(28) $$\mathrm{Ext}^1_{GF(p)[G]}(I_p,I_p) \simeq (\Omega\mathbf{Z}/(\Omega\mathbf{Z})^2 \otimes_{\mathbf{Z}} GF(p), I_p)^{GF(p)[G]} \simeq \mathrm{Hom}(G,\mathbf{Z}_p)$$

as $\Omega\mathbf{Z}/(\Omega\mathbf{Z})^2 \simeq G/G'$.

<u>Example 2</u>. Set $\Omega_p I = \Omega\mathbf{Z} \otimes_{GF(p)} F$. Then it is not too difficult to prove that

(29) $$F[G]/(\Omega_p 1)^n$$

is the maximal factor module of F[G] of Loewy length at most n with all composition factors isomorphic to I.

Example 3. Let F be a field of characteristic 2. The reader is urged to check that the following Loewy series of $F[D]_{F[D]}$ are correct:

(30) D :

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$D_8, Q_8, \mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
I	I	I
II	II	III
I	II	III
	II	I
	I	

This may be obtained using Corollary 10.12, which provides the dimension of $F[D]/J(F[D])^2$.

For P a p.i.m. of some group algebra, let H(P), the heart of P, denote the unique maximal submodule modulo the unique minimal submodule. In the examples above, we saw that D_8, Q_8 and $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ have identical Loewy series. However, it may be proved that the heart of D_8 is isomorphic to the direct sum of two uniserial modules of Loewy length 3, while the hearts of Q_8 and $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ are indecomposable. It has been proved in general that if $P = F[G]_{F[G]}$ for G a p-group, F a field of characteristic p, then H(P) is indecomposable unless G is a dihedral 2-group, by P. Webb (1982). But we have to dig deeper in order to distinguish between Q_8 and $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

11. Orders.

For completeness, we briefly discuss the theory of $\mathfrak{p}$-adic rings:

Let R_0 be an integral domain and $\mathfrak{p}$ a prime ideal in R_0 such that $\bigcap_{i=1}^{\infty} \mathfrak{p}^i = 0$. This gives rise to the sequence

$$(1) \qquad \cdots R_0/\mathfrak{p}^n \xrightarrow{\phi_n} R_0/\mathfrak{p}^{n-1} \to \cdots \to R_0/\mathfrak{p}^2 \xrightarrow{\phi_2} R_0/\mathfrak{p} \to 0$$

where, for $n \geq 2$ and $a \in R_0$, ϕ_n is defined by $\phi_n(a+\mathfrak{p}^n) = a + \mathfrak{p}^{n-1}$.

Now we take the inverse or projective limit $\varprojlim R_0/\mathfrak{p}^n$. We may think of this as the set

(2) $$\{(\ldots,a_n,a_{n-1},\ldots,a_2,a_1) \mid a_n \in R/\mathfrak{p}^n,\ \phi_n(a_n) = a_{n-1}\}$$

with coordinate addition and multiplication. The ring is called the $\mathfrak{p}$-adic completion of R_0 and will be denoted by R. We note that if $\mathfrak{q}$ is another ideal of R_0 such that $\mathfrak{q} \subseteq \mathfrak{p}$ while $\mathfrak{p}^r \subseteq \mathfrak{q}$ for some r, then $\varprojlim R_0/\mathfrak{p}^i = \lim R_0/\mathfrak{q}^i$.

There is a natural embedding of R_0 into R, namely

(3) $$x \to (\ldots,x+\mathfrak{p}^n,\ldots,x+\mathfrak{p}^2,x+\mathfrak{p})$$

and we identify x with its image in R.

Assume now furthermore that $\mathfrak{p} = (\pi)$ is a principal ideal. Then $R_0/\mathfrak{p}^n = R/R\pi^n$ and thus $R = \varprojlim R/R\pi^n$.

Finally, we assume $\mathfrak{p}$ is a maximal ideal in R_0. Let $x \in R_0/\mathfrak{p}^n$ such that $x \notin \mathfrak{p}/\mathfrak{p}^n$ for some n. As $R_0/\mathfrak{p}$ is a field, there exists $y \in R_0/\mathfrak{p}^n$ and $z \in \mathfrak{p}/\mathfrak{p}^n$ such that $xy = 1-z$. Hence

(4) $$xy(1 + z + z^2 + \ldots + z^{n-1}) \in 1 + \mathfrak{p}^n$$

and thus x is invertible in $R_0/\mathfrak{p}^n$. Consequently, an element of R is invertible if and only if it does not belong to $R\pi$. THus R is a principal ideal domain and local, and the ideals in R are precisely those of the form $R\pi^n$ and (0).

We call a ring R a p-<u>adic ring</u> if R is a P.I.D. of characteristic 0, has a unique maximal ideal (π) with $R/(\pi)$ of characteristic p and $R = \varprojlim R/(\pi^n)$.

Fields of characteristic p of course are p-adic rings, and we have just discussed how to construct such rings starting from integral domains, for instance the localization of the ring of algebraic integers in an algebraic number field at a prime ideal.

<u>Definition 11.1</u>. Let R be a commutative ring. By an R-<u>order</u> A, we understand an algebra over R, which is free and finitely generated as an R-module.

Let A be an R-order. As $A_R \simeq R^n$ for some n, it follows that if R is a p-adic ring, then $A = \varprojlim A/A(\pi^n)$, where (π) is the maximal ideal of R. This holds for A not only as an R-module, but as a ring as well. Also, we note that $A/A(\pi)$ is an algebra over the field $R/(\pi)$.

Example 1. Let R be a p-adic ring and G a finite group. Then the group ring $R[G]$ is an R-order. The center $Z(R[G])$ is an R-order as well, with basis $\{[K]\}$, where K runs through the conjugacy classes of G.

Example 2. Let A be an R-order, and M an (R-free and finitely generated, always) A-module. Then $\mathrm{End}_A(M)$ is an R-order.

In the following, we let R be a p-adic ring and A and R-order. Let (π) be the maximal ideal of R and set $F = R/(\pi)$. Thus $\overline{A} = A/A(\pi)$ is a finite dimensional algebra over F, and $\dim_F \overline{A} = \mathrm{rank}_R A$. We now define the radical of A, $J(A)$, by

$$(5) \qquad A/J(A) = \overline{A}/J(\overline{A}).$$

Now, as $J(\overline{A})$ is nilpotent, there exists a natural integer n with $J(A)^n \subseteq A(\pi)$. Conversely, $A(\pi) \subseteq J(A)$ by definition, and consequently, as remarked earlier,

$$(6) \qquad A = \varprojlim A/J(A)^i.$$

We may now take advantage of this to obtain

Theorem 11.2. Same notation as above. Then

i) Let $\overline{e} \in \overline{A}$ be an idempotent. Then there exists an idempotent $e \in A$ such that $\overline{e} = e + A(\pi)$.

ii) Let $\overline{1} = \sum_{i=1}^{t} \overline{e}_i$ be a primitive idempotent decomposition in $\overline{A}$. Then there exists a primitive idempotent decomposition $1 = \sum_{i=1}^{t} e_i$ in A such that $\overline{e}_i = e_i + A(\pi)$ for all i. Furthermore, if

$1 = \sum_{i=1}^{s} e_i'$ is another primitive idempotent decomposition in A, then $s = t$ and there exists a unit $u \in A$ such that $e_i' = u^{-1}e_iu$ for all i.

Proof: i) follows from ii). Now, by Theorem 1.5, an idempotent decomposition of $\bar{1}$ in $\bar{A}$ may successively (and succesfully) be lifted to one in $A/A(\pi^n)$. Choosing elements of A with all these as their respective projections, for all n, this gives an idempotent decomposition in A of 1. Moreover, if $e \in A$ is an idempotent, eAe is an R-order, and the argument we just gave shows that e is primitive if and only if $e + eAe(\pi)$ is primitive in $eAe/eAe(\pi)$, thereby proving the first part of i). Next we use the fact that $\bar{A}$ is artinian to deduce that as $1 = \sum_{i=1}^{s} e_i'$ reduced modulo $A(\pi)$ is a primitive idempotent decomposition in $\bar{A}$, we necessarily have that $s = t$ by Theorem 1.5, which moreover yields the existence of a unit $\bar{u}_0 \in A$ such that $\bar{u}_0^{-1}\bar{e}_i\bar{u}_0 = \bar{e}_i'$ for all i with suitable notation. We now lift $\bar{u}_0$ to a unit of A just as we did with idempotents above. In particular, we may for the rest of the proof assume that $\bar{e}_i = \bar{e}_i'$ for all i. But again by Theorem 1.5, there exists $u_n \in A$ such that $u_n + A(\pi^n)$ is a unit in $A/A(\pi^n)$, say with inverse $v_n + A(\pi^n)$, and $v_ne_iu_n + A(\pi^n) = e_i' + A(\pi^n)$ for all i, and also $u_n+A(\pi^{n-1}) = u_{n-1}+A(\pi^{n-1})$ for all n. Consequently, $u = (u_n) \in A = \varprojlim A/A(\pi^n)$ will do as the claimed unit in ii).

Just as in the case of finite dimensional algebras, this enables us to prove Krull-Schmidt for A-modules, A as above, where again an A-module M is free and finitely generated as R-module.

Corollary 11.3 (Krull-Schmidt). Same notation as above. The indecomposable direct summands of M are uniquely determined up to isomorphism. In other words, if

$$M \simeq \bigoplus_{i \in I} M_i' \simeq \bigoplus_{j \in J} M_j'' \tag{7}$$

with M_i' and M_j'' indecomposable A-modules, then there exists a bijection $\phi : I \to J$ such that $M_i' \simeq M''_{\phi(i)}$ for all i.

Proof: By Theorem 5.2, Theorem 11.2 applied to $\mathrm{End}_A(M)$ and then by appealing to Theorem 1.4.

Corollary 11.4. M is indecomposable if and only if $\mathrm{End}_A(M)/J(\mathrm{End}_A(M))$ is a division algebra over F.

Proof: By Theorem 11.2 and Lemma 5.3.

Now, let $\overline{1} = \sum_{i=1}^{t} \overline{e}_i$ be a primitive idempotent decomposition of $\overline{1}$ in $\overline{A}$, and let $1 = \sum_{i=n}^{t} e_i$ be a lift of this to A. Set $\overline{P}_i = \overline{e}_i\overline{A}$ and $P_i = e_iA$. Then $\overline{P}_i \simeq P_i/P_i\pi$,

(8) $$A_A = \bigoplus_{i=1}^{t} P_i$$

and P_i is a p.i.m. of A. Moreover, just as in Theorem 3.14, we have

Corollary 11.5. The following are equivalent

i) There exists a unit $\overline{u} \in \overline{A}$ with $\overline{u}^{-1}\overline{e}_i\overline{u} = \overline{e}_j$.

ii) $\overline{P}_i \simeq \overline{P}_j$.

iii) There exists a unit $u \in A$ with $u^{-1}e_iu = e_j$.

iv) $P_i \simeq P_j$.

Corollary 11.6. Let $\overline{P}_1,\ldots,\overline{P}_\ell$ be a complete set of representatives of the p.i.m.'s of $\overline{A}$. Then $P_1,\ldots,P_\ell$ is a complete set of representatives of the p.i.m.'s of A.

Finally, Lemma 5.4 yields

Lemma 11.7. Let $e \in A$ be an idempotent and M and A-module. Then

(9) $$(eA,M)^A \simeq M_e.$$

Proof: Reduce modulo π (or use the same proof as in Lemma 11.7).

12. Modular systems and blocks.

What we have achieved in the previous section is a machinery, which allows us to go from characteristic 0 to characteristic p, and back again for certain modules. Let us formalize this:

Definition 12.1. Let R be a p-adic ring with maximal ideal (π). Let $F = R/(\pi)$, and let S denote the quotient field of R. Then (F,R,S) is called a p-modular system.

Example 1. Let K be an algebraic number field of characteristic 0, let R_0 denote the ring of algebraic integers of K and let p be a prime. Let $\mathfrak{p}_0$ be a maximal ideal of R_0 such that $\mathfrak{p}_0 \cap \mathbb{Z} = (p)$. Then the localization $R_{\mathfrak{p}_0}$ of R_0 at $\mathfrak{p}_0$ is a discrete valuation ring. Let $\mathfrak{p}$ denote the maximal ideal of $R_{\mathfrak{p}_0}$. We now form the completion $R = \varprojlim R_{\mathfrak{p}_0}/\mathfrak{p}^n$ and denote its maximal ideal by (π), in view of our analysis in the previous section. This will be the type of p-adic ring we are typically dealing with.

Let (F,R,S) be a p-modular system, and let A be an order over R of finite rank. Set $\tilde{A} = A \otimes_R S$ and $\overline{A} = A/A\pi$. Let $\phi : A \to \overline{A}$ be the canonical homomorphism. Thus we have

$$(1) \qquad 0 \leftarrow \overline{A} \overset{\phi}{\leftarrow} A \subseteq \tilde{A}$$

which induces

$$(2) \qquad Z(\overline{A}) \overset{\phi}{\leftarrow} Z(A) \subseteq Z(\tilde{A}).$$

Now let $\mathbb{B}_1,\ldots,\mathbb{B}_r$ be the blocks of A and denote the unity of $\mathbb{B}_i$ by ε_i. Then $1 = \sum_{i=1}^{r} \varepsilon_i$ is a primitive idempotent decomposition in $Z(\overline{A})$. However, our assumption does not allow us to deduce that $\phi(Z(A))$ equals $Z(\overline{A})$, although this is the case if A is a group algebra. Nevertheless, we have

Proposition 12.2. Let $\bar{\varepsilon} \in \bar{A}$ be a central primitive idempotent, and let ε be a lift of $\bar{\varepsilon}$ to A. Then ε is central in A. In particular, ε is uniquely determined.

Proof: (Dade (1973), Prop. 1.12). We have that

(3) $$A = \varepsilon A \varepsilon \oplus \varepsilon A(\varepsilon-1) \oplus (\varepsilon-1)A\varepsilon \oplus (\varepsilon-1)A(\varepsilon-1).$$

Moreover, as $\bar{\varepsilon}$ is central in $\bar{A}$, $\varepsilon A(\varepsilon-1)$ and $(\varepsilon-1)A\varepsilon$ are both contained in $A\pi$. Thus we have for all $a \in A$ that $\varepsilon a(\varepsilon-1) \in A\pi$ and hence that $\varepsilon a(\varepsilon-1) \in A\pi^n$ for all n as ε is an idempotent, which forces $\varepsilon a(\varepsilon-1) = 0$. Similarly, $(\varepsilon-1)a\varepsilon = 0$. Hence $a\varepsilon = \varepsilon a \varepsilon = \varepsilon a$, and thus ε is central. It now follows from Theorem 11.2ii) that ε is uniquely determined.

This allows us to extend the concept of a block: Let $\mathbb{B}_1,\ldots,\mathbb{B}_r$ be the blocks of $\bar{A}$ and denote the unity of $\mathbb{B}_i$ by $\bar{\varepsilon}_i$. Thus $1 = \sum_{i=1}^{r} \bar{\varepsilon}_i$ is a primitive idempotent decomposition in $Z(\bar{A})$.
Correspondingly, we have by Proposition 12.2 a primitive idempotent decomposition $1 = \sum_{i=1}^{r} \varepsilon_i$ in $Z(A)$, where $\varepsilon_i + A\pi = \bar{\varepsilon}_i$. If we set $\hat{\mathbb{B}}_i = \varepsilon_i A$, then

(4) $$A = \bigoplus_{i=1}^{r} \hat{\mathbb{B}}_i$$

and $\mathbb{B}_i = \hat{\mathbb{B}}_i/\hat{\mathbb{B}}_i\pi$. Also, $\varepsilon_i \in Z(\tilde{A})$ and

(5) $$\tilde{A} = \bigoplus_{i=1}^{r} \varepsilon_i\tilde{A}.$$

However, ε_i is not necessarily primitive in $Z(\tilde{A})$. If $\varepsilon_i = \sum_{j=1}^{k_i} e_{ij}$ is a primitive idempotent decomposition in $Z(\tilde{A})$, then

(6) $$\varepsilon_i\tilde{A} = \bigoplus_{j=1}^{k_i} e_{ij}\tilde{A}$$

and $e_{ij}\tilde{A}$ is a simple Wedderburn component of $\tilde{A}$ for all j if $\tilde{A}$ is semisimple.

Another useful and quite surprising result is the following.

Proposition 12.3. (Dade (1973), Proposition 1.9) Let A be an R-order of finite rank, and let A' be a subalgebra (with or without the same unity). Then $J(A) \cap A' \subseteq J(A')$.

The proof is based on the following.

Lemma 12.4. If x is a unit in A, then x^{-1} lies in $R[x]$.

Proof: Let $R[x]$ be of rank n over R. Then $R[x] = \mathrm{Span}_R\{1,x,x^2,\ldots,x^{n-1}\}$. Let T be the R-linear transformation of A defined by $y \to xy$. As A is an R-order, we can define the characteristic polynomial

$$f(X) = \det(X1-T) = X^n + a_1X^{n-1} + \ldots + a_n \tag{7}$$

where $a_i \in R$ for all i. As x is invertible where a_i R for all i. As x is invertible in A, $\det(T) = (-1)^n a_n$ is a unit in R. Thus

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \tag{8}$$

implies that

$$x(-a_n)^{-1}(x^{n-1} + a_1x^{n-2} + \ldots + a_{n+1}) = 1 \tag{9}$$

which proves the statement.

Proof of Proposition 12.3: Let e be the unity of A'. Then e is an idempotent of A, and thus eAe is a suborder of A containing A'. By definition of $J(A)$ and Lemma 5.6, $J(eAe) = eJ(A)e = J(A) \cap eAe$. Thus we may as well assume that $A = eAe$, i.e., that $e = 1$. Now, $J(A) \cap A'$ is a 2-sided ideal in A' and for all $y \in J(A) \cap A'$, $1 - y$ is a unit in A. Hence $1 - y$ is a unit in A' by Lemma 12.4 for all $y \in J(A) \cap A'$, and it follows that $J(A) \cap A' \subseteq J(A')$. Indeed, if not, there exists a maximal ideal M in A' not containing $J(A) \cap A'$. Thus $M + J(A) \cap A' = A'$. In particular, $1 = m+a$ with $m \in M$, $a \in J(A) \cap A'$. But then $m = 1-a$ is a unit in A', a contradiction.

Example 2. Let G be a finite group, and set $A = R[G]$. Let χ be an irreducible character of $S[G]$. Then $\chi(\varepsilon_i) \neq 0$ if and only if the Wedderburn component corresponding to χ occurs in (6).

Definition 12.5. Let (F,R,S) be a p-modular system, and let A be a finite dimensional algebra over R. Set $\tilde{A} = A \otimes_R S$ and $\overline{A} = A/A\pi$. A p-block of A (or sometimes just a block) is identified by a central primitive idempotent $\overline{\varepsilon}$ of $\overline{A}$. Given such an idempotent, let ε denote the corresponding idempotent of A. Then the block $\mathbb{B}(\varepsilon)$ consists of $\overline{\varepsilon A}$, εA and $\varepsilon\tilde{A}$ as well as any module for one of these rings, the corresponding representations and their traces.

If G is a finite group and $A = R[G]$, we will just write the p-blocks of G.

13. Centers.

For later applications, it is worthwhile at this stage to consider the centers of the various algebras involved and the relations between them. As an immediate reward, we obtain that the number of isomorphism classes of simple $F[G]$-modules for an arbitrary finite group G and a field F which is a splitting field of $F[G]$ modulo its radical, equals the number of conjugacy classes of elements of order prime to char(F) (Brauer (1935)).

First, however, we consider an arbitrary finite dimensional algebra A over some field F.

Lemma 13.1. $J(Z(A)) = Z(A) \cap J(A)$.

Proof: As $J_0 := Z(A) \cap J(A)$ is a nilpotent ideal of $Z(A)$, $J_0 \subseteq J(Z(A))$. On the other hand, if $z \in J(Z(A))$, then z is a central nilpotent element in A, whence generates a nilpotent ideal, which shows that $z \in J(A)$.

Corollary 13.2. Suppose F is a splitting field of $A/J(A)$. Then F is a splitting field of $\overline{Z} := Z(A)/J(Z(A))$ as well.

Proof: Set $\overline{A} = A/J(A)$. By Lemma 13.1, $\overline{Z}$ is a subring of $Z(\overline{A})$, all of whose Wedderburn components by assumption are isomorphic to F. As $\overline{Z}$ is a semisimple algebra over F, the same must hold for $\overline{Z}$.

For V resp. v a subspace resp. an element of A, we denote the image under the canonical map $A \to A/J(A)$ by $\overline{V}$ resp. $\overline{v}$.

Following R. Brauer (1956), we now define

(1) $$S(A) = \mathrm{span}_F\{ab-ba \mid a,b \in A\}.$$

It immediately follows from this definition that

(2) $$S(\overline{A}) = \overline{S(A)}.$$

Proposition 13.3. Assume F is a splitting field of $\overline{A}$. Then

i) $\overline{A} = Z(\overline{A}) \oplus S(\overline{A})$. In particular, the codimension of $S(A) + J(A)$ in A equals the number of isomorphism classes of simple A-modules.

ii) The number of blocks of A equals $\dim_F \overline{Z}$. In particular, $\overline{Z} = Z(\overline{A})$ if and only if each block of A contains exactly one isomorphism class of simple modules.

Proof: Let $\overline{A} = \bigoplus_{i=1}^{\ell} A_i$, where the A_i's are the Wedderburn components. Then

(3) $$S(\overline{A}) = \bigoplus_{i=1}^{\ell} S(A_i).$$

Choosing the standard basis of $A_i \simeq \mathrm{Mat}_{n_i}(F)$, it is easy to see that $S(A_i) = \{M \in A_i \mid Tr(M) = 0\}$ is of codimension 1 in A_i. Obviously, $Z(A_i) \simeq F$ is a complement, and i) follows.

ii) Let $1 = \sum_{i=1}^{r} e_i$, where the e_i's are the block idempotents of A. As $\overline{Z} \simeq Z(A)/J(Z(A))$ by Lemma 14.1, $\overline{1} = \sum_{i=1}^{r} \overline{e}_i$ is a primitive idempotent decomposition of $\overline{1}$ in $\overline{Z}$. However, as F is a splitting of $\overline{Z}$ by Corollary 14.2, we deduce that $\overline{Z} = \mathrm{span}_F\{\overline{e}_1,\ldots,\overline{e}_r\}$ proving the first statement. Now the second statement follows from the first and i).

Next we need a technical result.

Lemma 13.4. i) Let $a,b \in A$. Then

(4) $$(a+b)^{p^r} \equiv a^{p^r} + b^{p^r} \mod S(A)$$

for all r.

ii) Assume F is a splitting field of $A/J(A)$. Let $\phi : \overline{A} \to \overline{A}$ denote the map $a \to a^p$. Then $\phi(S(\overline{A})) \subseteq S(\overline{A})$ and ϕ induces an isomorphism on $\overline{A}/S(\overline{A})$.

iii) $S(A) + J(A) = \{a \in A \mid a^{p^i} \in S(A)$ for some $i \in \mathbb{N}\}$.

Proof: i) As $ab \equiv ba \mod S(A)$, $(a+b)^p \equiv a^p+b^p \mod S(A)$, and (4) follows by induction.

ii) It suffices to consider the case when $\overline{A}$ is simple, where it immediately follows, as $\overline{A} = FI \oplus S(\overline{A})$ where I is the identity matrix, and $S(\overline{A}) = \{M \in \overline{A} \mid \mathrm{Tr}M = 0\}$ as we saw in the proof of Proposition 14.3. This forces $\phi(S(\overline{A})) \subseteq S(\overline{A})$, while $\phi(FI) = FI$.

iii) follows from ii).

For more results along this line, we refer the reader to Külshammer (1981).

We now return to group algebras. Let G be an arbitrary finite group and $g_1,\ldots,g_k$ representatives of the conjugacy classes of G. Denote the conjugacy class containing g_i by K_i and define $[K_i] = \sum_{g \in K_i} g$. First we observe

Lemma 13.5. Let Θ be an arbitrary commutative ring. Then

(5) $$Z(\Theta[G]) = \mathrm{span}_\Theta\{[K_1],\ldots,[K_k]\}.$$

Proof: Easy exercise.

Again, let (F,R,S) be a p-modular system and denote the maximal ideal of R by (π). By Lemma 13.5, we have that

(6) $$Z(R[G]) \otimes_R S \simeq Z(S[G])$$

and

(7) $$Z(R[G])/(\pi)Z(R[G]) \simeq Z(F[G]).$$

At this stage we may as well present a useful criterion due to R. Brauer, which indicates precisely when two irreducible characters lie in the same p-block. In particular, this is completely decided from the character table.

Recall that if S is a splitting field, then any irreducible character χ defines a central character $\omega_i : Z(S[G]) \to S$ given by

(8) $$\omega_i(a) = \frac{\chi(a)}{\chi(1)} .$$

Theorem 13.6. Let χ_1 and χ_2 be irreducible characters of G. Then χ_1 and χ_2 lie in the same block of G if and only if

(9) $$\omega_1([K]) \equiv \omega_2([K]) \bmod(\pi)$$

for any conjugacy class K of G.

Proof: This is an easy consequence of the fact that $\overline{Z} = (Z(F[G]) + J(F[G]))/J(F[G]) = \mathrm{Span}_F\{\overline{e}_1,\ldots,\overline{e}_r\}$ where $e_1,\ldots,e_r$ are the block idempotents of $F[G]$ and $\overline{e}_i = e_i + J(F[G])$, as we saw in Proposition 13.3. We recall that $\omega_i([K])$ is an algebraic integer. Hence ω_i maps $Z(R[G])$ onto R and $(\pi)Z(R[G])$ onto (π). Consequently, the induced map $\overline{\omega}_i([K]) = \omega_i([K]) + (\pi)$ is a representation of $Z(F[G]) \to F$ with $J(Z(F[G]))$ contained in its kernel. It now follows from the structure of $\overline{Z}$ that the induced representation $\overline{\omega}_i : \overline{Z} \to F$ satisfies

(10) $$\overline{\omega}_i(\overline{e}_j) = \begin{cases} 1 & \text{if } \omega_i \in \mathbb{B}_j \\ 0 & \text{if } \omega_i \notin \mathbb{B}_j \end{cases}$$

where $\mathbb{B}_j = e_jF[G]$.

Lemma 13.7. Set $A = F[G]$, and assume F is a splitting field of $A/J(A)$. Then

(11) $$S(A) = \{\Sigma\alpha_g g \mid \sum_{g \in K_i} \alpha_g = 0 \text{ for all } i=1,\ldots,k\}.$$

Proof: It easily follows from the definition of $S(A)$ that $S(A)$ is spanned by elements of the form $xy-yx$, where $x,y \in G$. As xy and yx lie in the same conjugacy class, one inclusion above is established. Conversely, as $x^{-1}g_i x \equiv g_i \mod S(A)$ for all $i=1,\ldots,k$ and all $x \in G$, we have that

(12) $$\sum_{g \in K_i} \alpha_g g \equiv (\sum_{g \in K_i} \alpha_g) g_i \mod S(A)$$

which proves the other inclusion.

An element in G is called p-regular, if its order is prime to p, otherwise p-singular. An arbitrary element $g \in G$ may be written uniquely as $g = g_p g_i'$ where g_p is a p-element, g' is p-regular and g_p and g' commute, as is easily established by considering the group $\langle g \rangle$. This is called the p-decomposition of g.

Theorem 13.8 (Brauer (1935)). Let G be a finite group and F a splitting field of $F[G]/J(F[G])$ of characteristic $p > 0$. Then the number of isomorphism classes of simple $F[G]$-module equals the number of conjugacy classes of p-regular elements.

Proof: Set $A = F[G]$ and let $g \in G$ be arbitrary, with p-decomposition $g = g_p g'$. Choose $n \in \mathbb{N}$ with $g^{p^n} = g'$. Then

(13) $$\begin{aligned}(g-g')^{p^n} &\equiv g^{p^n} - (g')^{p^n} \mod S(A)\\ &\equiv 0 \mod S(A)\end{aligned}$$

by Lemma 13.4i). Hence $g-g' \in S(A) + J(A)$ by Lemma 13.4iii). Let $g_1,\ldots,g_\ell$ denote the p-regular elements among $g_1,\ldots,g_k$, with suitable notation. Again, as $x^{-1}g_i x \equiv g_i \mod S(A)$ for all $x \in G$ and arbitrary i, the fact that $g \equiv g' \mod(S(A) + J(A))$ shows that

$$(14) \qquad \bar{A}/S(\bar{A}) \simeq \mathrm{Span}_F\{g_i + S(A) + J(A) \mid i=1,2,\dots,\ell\}$$

in $A/(S(A) + J(A))$, where $\bar{A} = A/J(A)$. Thus it remains to show that these elements are linearly independent. Assume therefore that

$$(15) \qquad \sum_{i=1}^{\ell} \alpha_i g_i \in S(A) + J(A).$$

Then Lemma 13.4iii) asserts the existence of an n such that $(\sum_{i=1}^{\ell} \alpha_i g_i)^{p^n} \in S(A)$. But then Lemma 13.4i) yields that

$$(16) \qquad \sum_{i=1}^{\ell} \alpha_i^{p^n} g_i^{p^n} \in S(A).$$

However, as $g_1,\dots,g_\ell$ form a set of representatives of the p-regular conjugacy classes, so do $g_1^{p^n},\dots,g_\ell^{p^n}$. Hence $\alpha_i = 0$ for all i by Lemma 13.7 and we are done.

14. R-forms and liftable modules.

Let (F,R,S) be a p-modular system, let (π) be the maximal ideal of R and let A be a finite dimensional algebra over R. Set $\tilde{A} = A \otimes_R S$ and $\bar{A} = A/A\pi$.

Before we start, let us point out an important consequence of the fact that R in particular is a P.I.D. Namely, if M is an R-free A-module, then any A-submodule of M is R-free, too. This, of course, is not usually the case with factor modules (thus $\bar{A}$ is a torsion module over R). For this reason, an A-module will for the rest of this book, unless otherwise stated, mean a module which, considered as an R-module, is free (and finitely generated, of course).

Definition 14.1. Let M be an $\tilde{A}$-module. By an R-form of M we understand an R-free A-module $\hat{M}$ such that $M \simeq \hat{M} \otimes_R S$.

Lemma 14.2. Any $\tilde{A}$-module has an R-form and if $\hat{M}$ is an R-form of M, then $\mathrm{rank}_R \hat{M} = \dim_S M$.

Proof: Let $\{m_1,\dots,m_s\}$ be an S-basis of M. As $\hat{M}$ we then choose the A-module spanned by $\{m_1,\dots,m_s\}$, to obtain an R-form of M. On the other hand, if X is an arbitrary R-free A-module, then $\dim_S(X \otimes_R S) = \operatorname{rank}_R(X)$, from which the second part follows.

Given any $\tilde{A}$-module M and any R-form $\hat{M}$ of M, we may consider the $\bar{A}$-module $\bar{\hat{M}} := \hat{M}/\hat{M}\pi$. It is usually possible to choose different non-isomorphic R-forms of the same $\tilde{A}$-module M as we shall see. Nevertheless, it is easy to see that the multiplicities of the simple $\bar{A}$-modules of $\bar{\hat{M}}$ are independent of the choice of $\hat{M}$, as we will experience in the next section.

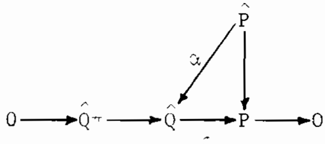
Definition 14.3. An $\bar{A}$-module L is called liftable if there exists an A-module $\hat{L}$ such that $\hat{L}/\hat{L}\pi \simeq L$. We call $\hat{L}$ a lift of L.

In particular, as we have seen, all p.i.m.'s of $\bar{A}$ are liftable. Also we stress the fact that in general, lifts are not uniquely determined. However,

Lemma 14.4. Let P be a p.i.m. of $\bar{A}$. Then lifts of P are uniquely determined up to isomorphism and, in particular, are p.i.m.'s of A.

Proof: P may be lifted to a projective A-module $\hat{P}$. Suppose $\hat{Q}$ is another lift. As $\hat{P}$ is projective, there exists $\alpha : \hat{P} \to \hat{Q}$ with

(1)



As $\operatorname{Ker} \phi = \hat{Q}\pi$, α must be surjective. As $\hat{Q}$ is R-free, α is injective as well, and thus $\hat{P} \simeq \hat{Q}$.

If $A = R[G]$ for some finite group G, it is possible to prove a more general statement than Lemma 14.4. Namely, if $H \leq G$ and I_H is the trivial $F[H]$-module, then any direct summand of $I_H^{\uparrow G}$ is liftable. This will be proved in Chapter II, Section 7.

Let us return to our general setup. Recall that if A is a ring and X and Y are A-modules, then $\mathrm{Hom}_A(X,Y)$ is denoted by $(X,Y)^A$.

Lemma 14.5. Let M_i, $i=1,2$, be arbitrary A-modules. Let $\overline{M}_i = M_i/M_i\pi$ and $\tilde{M}_i = M_i \otimes_R S$. Then

(2) $$(M_1,M_2)^A/(M_1,M_2)^A\pi \subseteq (\overline{M}_1,\overline{M}_2)^{\overline{A}}$$

(3) $$(M_1,M_2)^A \otimes_R S \simeq (\tilde{M}_1,\tilde{M}_2)^{\tilde{A}}.$$

In particular, if G is a finite group and $A = R[G]$, and S is a splitting field of $S[G]$,

(4) $$\mathrm{rank}_R((M_1,M_2)^{R[G]}) = (\chi_{\tilde{M}_1},\chi_{\tilde{M}_2})_G$$

where $\chi_{\tilde{M}_i}$ is the character of $\tilde{M}_i$.

Proof: (2) is trivial, while (3) follows from the following observation: As M_i is naturally embedded into $\tilde{M}_i$, $(M_1,M_2)^A$ is naturally embedded into $(\tilde{M}_1,\tilde{M}_2)^{\tilde{A}}$. Choose α arbitrary in the latter. As any element of S is of the form $r\pi^{-n}$ for some unit $r \in R$ and some $n \in \mathbb{N}$, we see by applying α to some basis of M_1 that $\pi^{-m}\alpha \in (M_1,M_2)^A$ for sufficiently large m.

Warning: Usually, equality does not hold in (2).

Definition 14.6. Same notation as above. Maps of $(\overline{M}_1,\overline{M}_2)^{\overline{A}}$ lying in $(M_1,M_2)^A/(M_1,M_2)^A\pi$ are called liftable.

It is usually very difficult to prove that a homomorphism is liftable, except if one of the modules involved is projective.

Theorem 14.7. Let M be a liftable $\overline{A}$-module and P a projective $F[G]$-module. Then all maps in $(P,M)^{\overline{A}}$ are liftable.

If $A = R[G]$ for some finite group G, then all maps of $(M,P)^{\overline{A}}$ are liftable as well.

<u>Proof</u>: Let $\hat{M}$ be a lift of M, $\hat{P}$ the lift of P. Then Lemmas 5.4 and 11.7 combined yield the first statement, while the second follows by duality.

<u>Remark</u>: For modules over group algebras, this is actually a special case of a more general result, which will be proved in Chapter II, Section 6.

<u>Remark</u>. There is at least one more general and very important case, in which homomorphisms of F[G]-modules are liftable, namely if the modules involved are direct summands of permutation modules. This will be discussed in Chapter II, Section 7.

To summarize the results of Lemma 14.5 and Theorem 14.7, we have

<u>Proposition 14.8</u>. Let $A = R[G]$ for some finite group G. Let M_1, M_2 be A-modules, and set $\overline{M}_i = M_i/M_i^-$ for $i = 1,2$. Assume furthermore that S is a splitting field of $S[G]$. Then all maps in $(\overline{M}_1,\overline{M}_2)^{\overline{A}}$ are liftable if and only if

$$\dim_F(\overline{M}_1,\overline{M}_2)^{\overline{A}} = (\chi_1,\chi_2)_G \tag{5}$$

where χ_i is the characters of $M_i \otimes_R S$.

<u>Corollary 14.9</u>. Suppose S is a splitting field of $S[G]$. Then F is a splitting field of $F[G]/J(F]G])$.

<u>Proof</u>: With the notation above, F is a splitting field of $\overline{A}/J(\overline{A})$ if and only if $(E,E)^{\overline{A}} \simeq F$ for any simple $\overline{A}$-module E, that is if and only if the multiplicity of E in its projective cover P_E equals $\dim_F((P_E,P_E)^{\overline{A}})$ by the remark following Corollary 5.9. However, this number is independent of F as long as S is a splitting of $S[G]$ by Proposition 14.8.

If we still assume that $A = R[G]$ for some finite group G, it is possible to generalize Lemma 14.4. But first, we give a formal proof of the fact mentioned in the remark closing Section 6.

Theorem 14.10. Let P be a projective $R[G]$-module. Consider an exact sequence of $R[G]$-modules

(6) $$0 \to P \xrightarrow{\phi} M \to X \to 0.$$

In particular, we assume that $M/\phi(P)$ is R-free, or in other words, $\phi(P)$ is a direct summand of M, considered as an R-module. Then (6) splits.

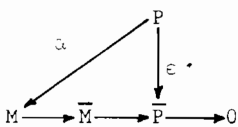
Proof: By assumption, the induced dual sequence is exact, too,

(7) $$0 \leftarrow P^{*} \leftarrow M^{*} \leftarrow X^{*} \leftarrow 0.$$

However, as evidently $P^{*}/P^{*-} \simeq (P/P\pi)^{*}$, which is projective, we have that P^{*} is projective and therefore (7) splits. Hence (6) split as well.

Corollary 14.11. Let M be an $R[G]$-module and set $\overline{M} = M/M\pi$. Let P be a p.i.m. of $R[G]$, and $\overline{P}$ the corresponding p.i.m. of $F[G]$. Then $\overline{P}$ is a direct summand of $\overline{M}$ if and only if P is a direct summand of M.

Proof: One way is trivial. Conversely, if $\overline{M} \simeq \overline{P} \oplus X$ for some $F[G]$-module X, it suffices to prove that a sequence of the form (6) exists. To obtain this, we consider

(8)


where α exists as P is projective. Moreover, as $\operatorname{Ker} \varepsilon = P\pi$, $M^{-} \cap \alpha(P) = \alpha(P)^{-}$, which proves that $\alpha(P)$ is an R-direct summand of M (see Definition 17.1).

15. Decomposition numbers and Brauer characters.

Let (F,R,S) be a p-modular system, and let A be a finite dimensional algebra over R. Set $\overline{A} = A/A\pi$ where (π) is the maximal ideal of A and $A = \tilde{A} \otimes_R S$.

We have seen how not only $\bar{A}$, but A as well is a direct summand of its indecomposable 2-sided ideals. As mentioned in the previous section, we will in due time prove (and in fact, the proof is very easy) that if G is a finite group and M is a direct summand of a permutation module of $F[G]$, then $\mathrm{End}_{F[G]}(M)$ is liftable. So the class of algebras we want to consider in the following, will at least include all endomorphism rings of direct summands of permutation modules of finite groups.

We recall that an A-module is supposed to be R-free (and finitely generated). The key to our discussion is the following fact.

<u>Theorem 15.1</u>. Let X and Y be A-modules such that $X \otimes_R S \simeq Y \otimes_R S$. Then any simple $\bar{A}$-module occurs with the same multiplicity in $X/X\pi$ and $Y/Y\pi$.

To prove this, we first point out an obvious but very important fact:

<u>Lemma 15.2</u>. Let M be an $\tilde{A}$-module, and let $\hat{M}_1$ and $\hat{M}_2$ be R-forms of M. Then there exists an injective A-homomorphism $\varphi : \hat{M}_1 \to \hat{M}_2$.

<u>Proof</u>: We may as well assume that

(1) $$\hat{M}_1 \subseteq \hat{M}_2 \otimes S \simeq \hat{M}_1 \otimes S.$$

In particular, for any $v \in \hat{M}_2 \otimes S$, there exists $n \in \mathbb{N}$ such that $v\pi^n \in \hat{M}_2 \otimes 1$. In particular, $\hat{M}_1\pi^m \subseteq \hat{M}_2 \otimes 1$ for sufficient large m, from which the lemma follows.

<u>Proof of Theorem 15.1</u> (Serre (1967), III.2.2). By Lemma 15.2, we may assume that $X \subseteq Y$. Hence there exists $r \in \mathbb{N}$ with $Y\pi^r \subseteq X$. We now apply induction to r: Assume $r = 1$. Then there exists an exact sequence

(2) $$0 \to Y\pi/X\pi \to X/X\pi \to X/Y\pi \to 0$$

(3) $$0 \to X/Y\pi \to Y/X\pi \to Y/X \to 0.$$

However, as $Y/X \simeq Y\pi/X\pi$ as A-modules, Theorem 15.1 holds in this case.

Next assume $r > 1$. Set $Z = \pi^{r-1}Y + X$. Then $\pi^{r-1}Y \subseteq Z \subseteq Y$, and $Z\pi \subseteq X \subseteq Z$. So by induction, the theorem holds for the pairs X and Z, Z and Y, and hence for X and Y.

We now impose an assumption on A, namely that $\tilde{A}$ is semisimple.

Let $M_1,\ldots,M_k$ be a full set of representatives of the isomorphism classes of simple $\tilde{A}$-modules, and $E_1,\ldots,E_\ell$ likewise of the simple $\bar{A}$-modules. Denote the projective cover of E by P_i and let $\hat{P}_i$ be the lift to A of P_i.

<u>Definition 15.3</u>. Assume $\tilde{A}$ is semisimple, and use the notation above. The <u>decomposition numbers</u> d_{ij}, $i=1,\ldots,k$, $j=1,\ldots,\ell$ are defined as the multiplicity of E_j as a composition factor of any R-form of M_i reduced modulo (π). Note that thus d_{ij} depend on R.

<u>Lemma 15.4</u>. Same notation and assumption as in Definition 15.3. Assume furthermore that S is a splitting field of $\tilde{A}$ and that F is a splitting field of $\bar{A}/J(\bar{A})$. Then

$$(4) \qquad \hat{P}_j \otimes_R S \simeq \bigoplus_{i=1}^{k} M_i^{(d_{ij})}.$$

<u>Proof</u>: Let $\hat{M}_i$ be an R-form of M_i. By Definition 15.3, Theorem 14.7 and Lemma 14.5,

$$(5) \quad d_{ij} = \dim_F((P_j,M_i/M_i\pi)^{\bar{A}} = \operatorname{rank}_R((\hat{P}_j,\hat{M}_i)^A) = \dim_S((\hat{P}_j \otimes S,\hat{M}_i)^{\tilde{A}})$$

from which (4) follows as $\tilde{A}$ is assumed to be semisimple.

It follows from the definition of a block of A that $d_{ij} = 0$ if M_i and E_j do not belong to the same block. Each block $\mathbb{B}_s$ defines a <u>decomposition matrix</u> $D_s = \{d^s_{ij}\}$, where M_{s_i} and E_{s_j} run through the simple $\tilde{A}$- resp. $\bar{A}$-modules of $\mathbb{B}_s$. By the decomposition matrix of G, we mean

$$(6)\qquad D = \begin{Bmatrix} D_1 & 0 & & 0 \\ 0 & D_2 & & \\ & & \ddots & \\ 0 & & & D_r \end{Bmatrix}$$

if G has r blocks.

We notice that D_s is indecomposable, i.e., for no rearrangement of rows and columns is it possible to write D_s as

$$(7)\qquad \begin{Bmatrix} D'_s & 0 \\ 0 & D''_s \end{Bmatrix}$$

again by the definition of a block.

<u>Theorem 15.5</u>. Same notation and assumption as in Definition 15.3. Assume further that F is a splitting field of $\bar{A}$, and let C denote the Cartan matrix of $\bar{A}$. Then C is symmetric. Assume furthermore that S is a splitting field of $\tilde{A}$, and let D^t denote the transpose of D. Then

$$(8)\qquad D^t D = C.$$

<u>Proof</u>: By Corollary 5.9, Lemma 14.5 and the fact that $\tilde{A}$ is semisimple,

$$(9)\qquad c_{ij} = \dim_s((P_j \otimes S, P_i \otimes S)^{\tilde{A}}) = \dim_s((P_i \otimes S, P_j \otimes S)^{\tilde{A}}) = c_{ji}.$$

If furthermore S is a splitting field of $\tilde{A}$, we moreover have that

$$(9)\qquad c_{ij} = \Sigma d_{sj} d_{si}$$

by Lemma 15.4.

<u>Remark</u>. It is <u>not</u> always true that C <u>is</u> symmetric, if F is not a splitting field of $\bar{A}$.

From here on, we will assume that $A = R[G]$ for some finite group G. As we have seen, certain problems in characteristic p may be transformed to problems in characteristic 0, and hopefully the rich theory of characters may then help us. To take advantage of this, we therefore introduce the concept of a Brauer character, which is based on the following observations:

As R is local, the restriction of the canonical homomorphism $R \to R/(\pi) = F$ to the set of units R^* in R is an isomorphism onto $F^{\#} = F\setminus\{0\}$. Denote the inverse of this by $\hat{\ }$.

In the following, we assume S is a splitting field for all subgroups of G. Let M be an arbitrary $F[G]$-module and $x \in G$ any p-regular element. Then $M_{\downarrow\langle x\rangle}$ is semisimple, and we may choose a basis for M so that the matrix of x w.r.t. this basis is a diagonal matrix $\{x_{ij}\}$.

With this notation, we now introduce

__Definition 15.6.__ By the __Brauer character__ ϕ_M of M, we understand the function $\phi_M : G_0 \to S$, where G_0 is the set of p-regular elements in G, defined by

(10) $$\phi_M(x) = \sum_i \hat{x}_{ii} \ .$$

In particular, $\phi_M(x)$ is an algebraic integer, ϕ_M is a class function, and we note that Brauer characters are just ordinary characters if p does not divide $|G|$. Also given an exact sequence

(11) $$0 \to M' \to M \to M'' \to 0$$

of $F[G]$-modules, we have that

(12) $$\phi_M = \phi_{M'} + \phi_{M''} \ .$$

Moreover, if M^* denotes the dual module of M as usual, $\phi_{M^*}(x) = \phi_M(x^{-1}) = \overline{\phi_M(x)}$, and thus ϕ_M and ϕ_{M^*} are complex conjugate. However,

__Warning.__ An algebraic conjugate of a Brauer character is not necessarily a Brauer character. Without going into detail, we just mention that A_8 provides a counterexample in characteristic 2. However, if two F[G]-modules are algebraically conjugate, then so are their Brauer characters.

Recall that (Corollary 14.9) as S is a splitting field of S[G], F is a splitting field of $F[G]/J(F[G])$ and thus $F[G]$ has ℓ different simple modules E_j, where ℓ is the number of conjugacy classes of elements of order prime to p by Theorem 13.8. The Brauer characters $\{\phi_j\}$ of these modules are called the <u>irreducible Brauer characters</u>.

Let $\chi_1,\ldots,\chi_k$ denote the irreducible characters of $S[G]$. The projective $F[G]$-cover P_j of E_j lifts to the $R[G]$-module $\hat{P}_j$, and the character of $\hat{P}_j \otimes_R S$ will be denoted by Φ_j. Finally, the decomposition matrix is denoted by $D = \{d_{ij}\}$ and the Cartan matrix by $C = \{c_{st}\}$. We then have the following relations.

<u>Lemma 15.7</u>. At elements of order prime to p, we have

i) $\chi_i = \sum_j d_{ij}\phi_j$

ii) $\Phi_s = \sum_t c_{st}\phi_t$

while at all elements in G,

iii) $\Phi_s = \sum_i d_{is}\chi_i$.

In particular, $(\Phi_s,\phi_t)_G = c_{st}$.

<u>Proof</u>: i) and ii) are by definition, while iii) follows from Lemma 15.4.

For $X \subseteq G$ a union of conjugacy classes, let $\mathrm{Char}_S(X)$ denote the vector space over S of class functions from X into S. We then define a (non-singular) inner product on $\mathrm{Char}_S(X)$ by

$$(13)\qquad (\eta_1,\eta_2)_X = \frac{1}{|G|}\sum_{x\in X}\eta_1(x)\eta_2(x^{-1}).$$

Recall that G_0 denotes the set of p-regular elements of G.

We now make the following interesting observation.

<u>Proposition 15.8</u>. Let $x \in G$ be p-singular. Then $\Phi_s(x) = 0$ for all $s=1,\ldots,\ell$.

Proof: We may as well assume that $G = \langle g \rangle$ for some $g \in G$, as the restriction of a projective R[G]-module to a subgroup remains projective. Let $g = g'g_p$ be the p-decomposition of g. Then $\mathrm{ord}(g') = \ell$, and the irreducible Brauer characters of G are simply the irreducible characters of $\langle g' \rangle$. Moreover, as $F[\langle g' \rangle]$ is semisimple, $\Psi_s := \phi_s \uparrow_{\langle g' \rangle}^{\langle g \rangle}$ is the character of a direct summand Q_s of $A = R[\langle g \rangle]$. Hence,

$$(14) \qquad A_A \simeq \bigoplus_{s=1}^{\ell} \hat{P}_s \simeq \bigoplus_{s=1}^{\ell} \hat{Q}_s$$

and as $\Psi_s \downarrow_{\langle g' \rangle} = \mathrm{ord}(g_p)\phi_s$, $c_{ss} \neq 0$ and the ϕ_s's clearly are linearly independent, it follows by Krull-Schmidt that $\Psi_s = \Phi_s$, which yields the statement.

Another way of stating Proposition 15.8 is, that $(\Phi_s,\Phi_t)_G = (\Phi_s,\Phi_t)_{G_0}$.

This allows us to supplement the basic Lemma 15.7 with the following omnibus theorem.

Theorem 15.9. Same notation as above.

i) $\{\phi_1,\ldots,\phi_\ell\}$ and $\{\Phi_1,\ldots,\Phi_\ell\}$ both form a basis for $\mathrm{Char}_S(G_0)$. In particular, the decomposition number and the Cartan invariants are uniquely determined by Lemma 15.7i) and ii).

ii) $\det C \neq 0$ and the rank of D is ℓ.

iii) a) $\{(\phi_s,\phi_t)_{G_0}\}_{s,t} = C^{-1}$

b) $(\Phi_s,\phi_t)_{G_0} = \delta_{st}$

c) Let $g_1,\ldots,g_\ell$ be representatives of the conjugacy classes of p-regular elements. Then

$$(15) \qquad \sum_j \Phi_j(g_{i_1})\phi_j(g_{i_2}^{-1}) = \delta_{i_1 i_2} \; |C_G(g_{i_1})|.$$

Proof: As the dimension of $\mathrm{Char}_S(G_0)$ is ℓ, i) will follow from Lemma 15.7ii) if and only if we can show that $\{\Phi_1,\ldots,\Phi_\ell\}$ spans $\mathrm{Char}_S(G_0)$. To see this, we first choose a natural basis of $\mathrm{Char}_S(G_0)$: For $i = 1,2,\ldots,\ell$, define

(16) $$\Lambda_i = \sum_{j=1}^{k} \chi_j(g_i^{-1})\chi_j .$$

Then $\Lambda_i(g_j) = \delta_{ij}|C_G(g_i)|$ and thus $\{\Lambda_1,\ldots,\Lambda_k\}$ spans $\mathrm{Char}_S(G_0)$. Now for i arbitrary, but fixed, let $\{\psi_s\}$ denote the set of irreducible characters of $<g_i>$. Now, if we set $\lambda_i = \sum_s \psi_s(g_i^{-1})\psi_s$, we observe that $\Lambda_i = \lambda_i^{\uparrow G}$. However, as $F[<g_i>]$ is semisimple $\psi_s^{\uparrow G}$ is the character of a direct summand of $R[G]$, i.e., of a projective module. Therefore $\psi_s^{\uparrow G}$ belongs to $\mathrm{Span}_S\{\Phi_1,\ldots,\Phi_\ell\}$, and consequently Λ_i does as well.

Now ii) follows from i) and Lemma 15.7.

By Proposition 15.8,

(20) $$C = (\Phi_s,\Phi_t)_G = (\Phi_s,\Phi_t)_{G_0} = C^2\Gamma$$

where $\Gamma = \{(\phi_s,\phi_t)_{G_0}\}_{s,t}$. Hence $\Gamma = C^{-1}$, as C is non-singular, which proves iii) a) and b). Finally, c) follows from the classical orthogonality relation:

(21) $$\begin{aligned}\delta_{i_1}\delta_{i_2}|C_G(g_{i_1})| &= \sum_i \chi_i(g_{i_1})\chi_i(g_{i_2}^{-1}) \\ &= \sum_i\sum_j\sum_s d_{is}\phi_s(g_{i_1})d_{ij}\phi_j(g_{i_2}^{-1}) \\ &= \sum_j\sum_s c_{js}\phi_s(g_{i_1})\phi_j(g_{i_2}^{-1}) \\ &= \sum_j \Phi_j(g_{i_1})\phi_j(g_{i_2}^{-1})\end{aligned}$$

16. Basic algebras and small blocks.

This section is partly inspired by some recent work by J. Brandt (1982). Our main concern will be to throw some light upon the relations between the number of irreducible characters and the number of irreducible Brauer characters in a block. In particular, we present a new and very short proof of the characterization of blocks with precisely one irreducible character, Brauer and Nesbitt (1941), without even mentioning what a defect group is. Our first observation is straightforward and goes by to Brauer and Nesbitt (1941).

Let G be a finite group and let (F,R,S) be a p-modular system such that S is a splitting field of $S[G]$. For $\mathbb{B}$ a block of $R[G]$, we let $k(\mathbb{B})$ denote the number of irreducible characters in $\mathbb{B}$ and $\ell(\mathbb{B})$ the number of irreducible Brauer characters in $\mathbb{B}$. We observe that as the decomposition matrix of $\mathbb{B}$ has rank $\ell(\mathbb{B})$, we must have $k(\mathbb{B}) \geq \ell(\mathbb{B})$.

Proposition 16.1. There is a one-to-one correspondence between blocks $\mathbb{B}$ of G with $k(\mathbb{B}) = 1$ and irreducible characters of G, whose degree is divisible by the order of a Sylow p-subgroup.

Proof: Let $|G| = p^a h$, where $(p,h) = 1$, and let χ be an irreducible character of G such that p^a divides $\chi(1)$. The corresponding central primitive idempotent of $S[G]$ has the form

(1) $$\varepsilon = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g.$$

However, $\frac{\chi(1)}{|G|} \in R$ by assumption, and thus $\varepsilon \in R[G]$ as $\chi(g)$ is an algebraic integer for all g. Hence ε must be a block idempotent in $R[G]$. In particular, $k(\mathbb{B}(\varepsilon)) = 1$.

Conversely, if χ is the only irreducible character of a block $\mathbb{B}$, then in particular, the character of the p.i.m. in $\mathbb{B}$ is a multiple of χ and consequently $\chi(x) = 0$ by Proposition 15.8. Hence p^a divides $\chi(1)$.

Corollary 16.2. Let $|G| = p^a h$ where $(p,h) = 1$. Let χ be an irreducible character of G. Then p^a divides $\chi(1)$ if and only if χ vanishes on every p-singular element.

Remark. We point out that the formulation of Corollary 16.2 does not require any concepts from modular representation theory. It is in fact possible to prove it in characteristic 0. See Feit (1967).

Let $\mathbb{B}(\varepsilon)$ be a block of $R[G]$, and let $\bar{\varepsilon}$ be the corresponding block idempotent of $F[G]$. Set $B = \bar{\varepsilon}F[G]$, the corresponding block of $F[G]$. Then

(2) $$k(\mathbb{B}(\varepsilon)) = \dim_F(Z(B))$$

as we saw in Section 13. So in order to find relations between the structure of B and the number of irreducible characters in the corresponding block of R[G], we apparently have to study Z(B).

In the following, we therefore let F be any field and B a finite dimensional algebra over F which is indecomposable as an algebra. Let $e_1,\dots,e_\ell$ be representatives of the conjugacy action of the units of B, which are mutually orthogonal. Then $e_0 = \sum_{i=1}^{\ell} \ell_i$ is an idempotent.

Definition 16.3. By the basic algebra B_0 of B, we understand

(2) $$B_0 = \mathrm{End}_B(e_0B).$$

Lemma 16.4. With the notation above, we have

i) $B_0 \simeq e_0Be_0$

ii) $Z(B_0) \simeq Z(B)$ as F-algebras.

Proof: i) is Corollary 5.5.

ii): Let $\phi : B \to B_0$ be the F-linear map $b \to e_0be_0$ for $b \in B$. We claim that the restriction of ϕ to Z(B) is a ring homomorphism onto $Z(B_0)$. Indeed, as $\phi(z) = e_0z = ze_0$ for $z \in Z(B)$, it already follows that ϕ is an algebra homomorphism. Assume next that $\phi(z) = 0$ for $z \in Z(B)$. Then

(3) $$zu^{-1}e_iu = u^{-1}ze_iu = 0$$

as well for all units in B and all $i=1,\dots,\ell(B)$, and consequently $ze = 0$ for any primitive idempotent e in B, which proves that $z = 0$. Finally, we must prove that $\phi(Z(B)) = Z(B_0)$. To see this, choose n so that B_B is isomorphic to a direct summand of $(e_0B)^n$, which is possible by our choice of e_0. Then $E := \mathrm{End}_B((e_0B)^n) \simeq \mathrm{Mat}_n(B_0)$, and consequently $Z(E) \simeq Z(B_0)$. However, using the successful argument above, we may now prove that Z(E) maps isomorphically into Z(B), and it follows that Z(B) and $Z(B_0)$ have the same dimension over F, which in turn forces $\phi(Z(B)) = Z(B_0)$.

Lemma 16.5.

(4) $$Z(e_0Be_0) \subseteq \bigoplus_{i=1}^{\ell} Z(e_iBe_i).$$

Proof: If $z \in Z(e_0Be_0)$, then $ze_i = e_ize_i$ for all i and thus $z \in \bigoplus_i e_iBe_i$, from which (4) easily follows.

Set $E_{ij} = (e_iB, e_jB)^B$. Then E_{ii} is a ring for all i, and $End_B(e_0B) \simeq \bigoplus_{i,j} E_{ij}$. In the following, we therefore identify these. If B is symmetric, we now have an important partial converse to Lemma 16.5:

Lemma 16.6. Let B be symmetric and let $\phi_i \in Z(E_{ii})$ with the property that all composition factors of $\phi_i(e_iB)$ are isomorphic to the socle of e_iB. Then $\phi_i \in Z(End_B(e_0B))$.

Proof: Let $\psi \in E_{rs}$. Then $\phi_i\psi = \psi\phi_i = 0$ by assumption unless $r = s = i$, where by assumption $\phi_i\psi = \psi\phi_i$. Now the assertion follows from Lemma 16.5.

Let us see how we can make use of this in a group algebra.

Theorem 16.7. Same notation and assumption as in the beginning of this section. Then one of the following always occur for a block $\mathbb{B}$ of $R[G]$, with corresponding block B of $F[G]$:

i) B is a simple ring

ii) $k(\mathbb{B}) \geq \ell(\mathbb{B}) + 1$.

Proof: (With the notation used in Definition 16.3). Let $\phi_i \in E_{ii}$ be a map whose kernel is the radical of e_iB, using the fact that B is symmetric. Also by our analysis in Section 5, $E_{ii} = J(E_{ii}) + F1$, where 1 is the identity of e_iB. Now, by choice of ϕ_i, $\phi_i\psi = \psi\phi_i = 0$ for all $\psi \in J(E_{ii})$, and thus $\phi_i \in Z(E_{ii})$. Hence $\phi_i \in Z(End_B(e_0B))$ by Lemma 16.6. Hence $k(\mathbb{B}) \geq \ell(\mathbb{B}) + 1$ unless the identity map of e_0B is spanned by the ϕ_i's, which forces $\ell = 1$, and ϕ_1 to be the identity of $E_{1,1}$, which is equivalent to B being a simple ring.

Corollary 16.8. The following are equivalent for a block $\mathbb{B}$ of $R[G]$ with corresponding block B of $F[G]$:

i) B is a simple ring

ii) $k(\mathbb{B}) = 1$

iii) The Cartan matrix of B is $\{1\}$.

Proof: Clear.

We now come to the main result of this section, which essentially is due to Brandt ((1982b), Thm. 2.6). Recall that the i'th socle of a module M is denoted by $S_i(M)$.

Theorem 16.9. Let A_0 be a symmetric algebra over F such that all simple modules of A_0 are 1-dimensional. Let $1 = \sum_{i=1}^{\ell} e_i$ be a primitive idempotent decomposition. Then

$$S_2(e_iB_0e_i) \subseteq Z(A_0) \tag{5}$$

for all $i=1,\ldots,\ell$.

Proof: As $e_iA_0e_i$ is symmetric (Theorem 7.6), $S_1(e_iA_0e_i) \subseteq S_1(e_iA_0)$, which is assumed to be 1-dimensional. Hence equality holds. Now choose $a \in S_2(e_iA_0e_i)$ and let $b \in e_iA_0e_i$ be arbitrary. As $e_iA_0e_i = J(e_iA_0e_i) \oplus Fe_i$, we may in fact choose $b \in J(e_iA_0e_i) = J(e_iA_0) \cap e_iA_0e_i$. By Lemma 8.3 and the fact that A_0 is symmetric, ab and ba both lie in $S_1(e_iA_0e_i)$. Let $0 \neq s \in S_1(e_iA_0e_i)$ and choose $f_1,f_2 \in F$ with $ab = f_1s$, $ba = f_2s$. Let λ be the symmetric form of A_0. Then

$$0 = \lambda(ab-ba) = (f_1-f_2)\lambda(s). \tag{6}$$

However, as Fs is an ideal of $e_iA_0e_i$, $\lambda(s) \neq 0$, and consequently $f_1 = f_2$, which proves that $ab = ba$.

Now, if we return to the situation we had in Definition 16.3 through Lemma 16.6, Theorem 16.9 yields

Corollary 16.10. Let $\phi_i \in E_{ii}$ with $\phi_i(e_iB)$ of Loewy length at most 2. Then $\phi_i \in Z(e_0Be_0)$.

Proof: If $\phi_i(e_iB)$ is of Loewy length at most 2, $\phi_i(e_i)J(B)e_i \subseteq S_1(e_iBe_i)$, which proves that $\phi_i(e_i) \in S_2(e_iBe_i)$.

Again, let us see how this applies to group algebras.

Corollary 16.11. Same notation and assumption as in the beginning of this section. Let $\mathbb{B}$ be a block of $R[G]$ and assume $k(\mathbb{B}) = 2$. Then $\ell(\mathbb{B}) = 1$, char $F = 2$, and the decomposition matrix of the corresponding block B of $F[G]$ is

(7) $$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Moreover, if E denotes the simple B-module and $n = \dim_F E$,

(8) $$B \simeq \mathrm{Mat}_n(F[\mathbf{Z}_2]).$$

Furthermore, if $|G| = 2^a h$, where h is odd, then $n = 2^{a-1}m$ where m is odd.

Proof: The first part of the statement is due to Brandt (1982). By Theorem 16.7, $\ell(\mathbb{B}) = 1$ and with the notation above, $b(\mathbb{B}) = 2 = \dim_F Z(e_0Be_0)$ forces $e_1Be_1 = S_2(e_1Be_1)$, i.e., e_1Be_1 and consequently e_1B is of Loewy length 2, as all composition factors of e_1B are isomorphic to E and e_1Be_1 is symmetric. Thus e_1Be_1 is a symmetric algebra of dimension 2, and

(9) $$B \simeq \mathrm{Mat}_n(e_1Be_1).$$

Let χ_1, χ_2 be the irreducible characters of B. Then $\chi_1 + \chi_2$ is the projective cover of E. In particular, p^a divides $\chi_1(1) + \chi_2(1)$ by Corollary 9.5, where $|G| = p^a h$ with $(p,h) = 1$. However, $\chi_1(1) = \chi_2(1) = n$, and by Proposition 16.1, $\chi_i(1)$ is not divisible by p^a. This forces $p = 2$ and thus $\chi_1(1) = 2^{a-1}m$ for m odd.

Finally, let $\alpha \in e_1 J(B) e_1$. Then $\alpha^2 = 0$, and thus $(1+\alpha)^2 = 1$, as $p = 2$. Hence $\langle\tau\rangle \simeq \mathbf{Z}_2$ where $\tau = 1+\alpha$, and $e_1 B e_1 \simeq F[\langle\tau\rangle]$.

Corollary 16.12. (Brandt (1982b)). Same notation as above. Let $\mathbb{B}$ be a block of $R[G]$, and let B be the corresponding block of $F[G]$. Assume $k(\mathbb{B}) > 2$, and let $E_1,\ldots,E_{\ell(\mathbb{B})}$ represent the isomorphism classes of simple B-modules. Then

$$(10) \qquad k(\mathbb{B}) \geq 1 + \ell(\mathbb{B}) + \sum_{i=1}^{\ell(\mathbb{B})} \dim_F \operatorname{Ext}^1_{F[G]}(E_i, E_i).$$

Proof: Note that if $\ell(\mathbb{B}) > 1$, then all p.i.m.'s in B have Loewy length at least 3, while if $\ell(\mathbb{B}) = 1$, the p.i.m. of B has Loewy length 2 if and only if $k(\mathbb{B}) = 2$ by Corollary 16.11. Thus all p.i.m.'s of B have Loewy length at least 3 by assumption, and (10) follows from Theorem 16.9, as the identity does not have an image of Loewy length at most 2.

Corollary 16.13. Let P be a p-group and F a field of characteristic 2. Then the penultimate radical of $F[P]$ is contained in $Z(F[P])$.

Remark. It is in fact not difficult to exhibit a basis of $S_2(F[P])$, which equals the penultimate radical of $F[P]$ as remarked in Section 10. As we saw in Corollary 10.11, the dimension of $S_2(F[P])$ is $n+1$, where $n = \operatorname{rank}(P/\Phi(P))$. Let $x_1,\ldots,x_n$ be generators of P. For each x_i, choose $P_i \trianglelefteq P$ of index p with $P = \langle P_i, x_i\rangle$. For $Q \leq P$, set $\Sigma(Q) = \sum_{g \in Q} g$. Then

$$(11) \qquad \{\Sigma(P),\ \Sigma(P_i)(1-x_i)^{p-2}\}$$

form a basis, as the reader is invited to check.

17. Pure submodules.

Definition 17.1. Let θ be a P.I.D. which is local with maximal ideal (π), and let M be a free θ-module.

An θ-submodule N of M is called (θ-) pure in M if one of the following equivalent conditions is satisfied

i) $N\pi = N \cap M\pi$

ii) N is a direct summand of M

iii) M/N is free

iv) $N + M\pi/M\pi \simeq N/N\pi$.

We leave it to the reader to verify that these conditions indeed are equivalent.

Lemma 17.2. Let M be a free θ-module, and let N_1 and N_2 be pure submodules. Then

i) $N_1 \cap N_2$ is θ-pure.

ii) Assume $N_1 \cap N_2 = 0$. Then $N_1 + N_2$ is pure if and only if

(1) $$(N_1+N_2)/(N_1+N_2) \cap M\pi \simeq N_1/N_1\pi \oplus N_2/N_2\pi.$$

Proof: Exercise.

Warning. It is not in general true that the sum of two pure submodules again is pure, obviously.

The reason for paying so much attention to this property is of course that if $N \subseteq M$ is pure, then the reduction of N modulo (π) is a submodule of the reduction of M modulo (π). In general, we only get a quotient of the reduction as a submodule.

We now return to a finite group G and a p-modular system (F,R,S).

Theorem 17.3 (Zassenhaus and others). Let M be an R-free $R[G]$-module, and let Φ denote the character of $M \otimes_R S$. Let $\Phi = \chi_1 + \chi_2$, where χ_i is a character. Then M contains an R-pure submodule N_i such that χ_i is the character of $N_i \otimes_R S$, $i = 1,2$.

Proof: Let $M \otimes_R S = V_1 \oplus V_2$, where χ_i is the character of V_i. Set $N_i = M \cap V_i$. Then N_i is an R-form of V_i, obviously. Now, as V_i is an S-space,

(2) $$N_i\pi = (M \cap V_i)\pi = M\pi \cap V_i = N_i \cap M\pi$$

and thus N_i is R-pure in M.

Corollary 17.4 (Thompson (1967)). Let Φ be the character of a p.i.m. of $R[G]$, and let E be the corresponding simple $F[G]$-module. Let $\Phi = \chi_1 + \chi_2$ where χ_i is a character. Then there exists an indecomposable R-form M of χ_1 such that $\mathrm{Soc}(M/M\pi) \simeq E$. In particular, $M/M\pi$ is indecomposable.

Proof: By Theorem 17.3, M may be choosen as an R-pure submodule of $\hat{P}$, where $\hat{P}$ is the p.i.m. with character Φ. Hence $M/M\pi$ is isomorphic to a submodule of $\hat{P}/\hat{P}\pi$, which has simple socle E.

Another way of phrasing this is

Corollary 17.5. Let χ be an irreducible character of G and let E be any simple $F[G]$-module, which occurs as a composition factor of an R-form of χ. Then an R-form M of χ may be chosen to satisfy $M/M\pi \simeq E$.

Corollary 17.4 is extremely useful. It was this result E. C. Dade needed to give a complete description of blocks with finitely many isomorphism classes of indecomposable modules. We will return to this in Chapter III.

Remark. Notice that with the notation of Theorem 17.3, M/N_1 resp. M/N_2 is R-free and in fact an R-form of χ_2 resp. χ_1. Consequently, we might as well have chosen R-forms in Corollaries 17.4 and 17.5 with simple heads isomorphic to E rather than socles.

We have also seen in Lemma 15.2 that arbitrary R-forms of the same $S[G]$-module may be embedded into each other. In particular, a projective $R[G]$-module is never injective. It is, however, as we saw in Section 14, injective in the category of R-free modules.

We end this section with

Lemma 17.7. Assume S is a splitting field of $S[G]$ and let χ be an irreducible character. Let M be an R-form of χ, and let N be an R-free R[G]-module. Then N is an R-form of χ if and only if N is isomorphic to an R[G]-submodule M' of M. Furthermore, $M' \simeq M$ as R[G]-module if and only if there exists an $n \in \mathbb{N}$ such that $M' = M\pi^n$.

Proof: As remarked above, N may be embedded into M if N is an R-form of χ by Lemma 15.2. Conversely, if $M' \subseteq M$ is an R[G]-submodule, then $M' \otimes_R S \simeq M \otimes_R S$, as $M \otimes_R S$ is simple, and thus M' is an R-form of χ.

Let $\phi : M' \to M$ be an isomorphism. Then $\phi \otimes 1 : M' \otimes_R S \to M \otimes_R S$ is an isomorphism as well, hence multiplication by a scalar from S, hence may be written as $r\pi^z$ where $z \in \mathbb{Z}$ and r is a unit in R. However, as $M' \subseteq M$, $z \leq 0$ and thus $M' = M\pi^n$ for $n = -z$.

18. Examples.

Example 1. $G = SL(2,4)$ in characteristic 2. Choose a 2-modular system (F,R,S) such that S is a splitting field of $S[G]$. We recall the character table of $SL(2,4) \simeq A_5$:

(1)

	1	2	3	5_1	5_2
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_4	5	1	-1	0	0
χ_5	4	0	1	-1	-1

where a conjugacy class is identified by the order of an element in it and, if necessary, an index. Thus $F[G]$ has 4 isomorphism classes of simple modules. As we have chosen G very wisely, we immediately see two non-trivial simple modules, 2_1 and 2_2 algebraically conjugate of dimension 2. So, together with the trivial module I, we know three of the four simples. However, by Corollary 16.8 and Proposition 16.1, we know that F[G] has exactly one block which is simple, with χ_5 as the character, and the simple module 4 of that block is of dimension 4. (Actually, $4 = 2_1 \otimes 2_2$, which is the so-called Steinberg module. Any group of Lie type has exactly one block in the describing characteristic which is simple, and the corresponding simple module is called the Steinberg module. For representations of groups of Lie type, we refer the reader to Curtis-Reiner (1985).

From Theorem 13.6 it follows that $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ form another block. Now, the small dimensions only leaves the following possibility for the decomposition matrix

(2)

	I	2_1	2_2	4
χ_1	1	0	0	0
χ_2	1	1	0	0
χ_3	1	0	1	0
χ_4	1	1	1	0
χ_5	0	0	0	1

and therefore, the Cartan matrix is

$$(3)\qquad \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, it still takes a lot of work to determine the Loewy series of the p.i.m.'s. In the following chapter, we will develop some methods which will make this very easy. So at this stage, we merely state the Loewy structure of the p.i.m.'s. The reader is of course more than welcome to try to work it out.

(4)

P_I	P_{2_1}	P_{2_2}	P_4
I	2_1	2_2	4
$2_1 2_2$	I	I	
I I	2_2	2_1	
$2_1 2_2$	I	I	
I	2_1	2_2	

and $F[G]_{F[G]} \simeq P_I \oplus P_{2_1}^{(2)} \oplus P_{2_1}^{(2)} \oplus 4^{(4)}$.

Moreover, (4) yields that we may choose R-forms of irreducible characters which reduced module 2 are indecomposable with the following Loewy structure, using Corollary 17.5

(5)

$$\chi_2 : \begin{matrix} I \\ 2_1 \end{matrix}, \quad \begin{matrix} 2_1 \\ I \end{matrix} \qquad\qquad \chi_3 : \begin{matrix} I \\ 2_2 \end{matrix}, \quad \begin{matrix} 2_2 \\ I \end{matrix}$$

$$\chi_4 : \begin{matrix} I \\ 2_1 2_2 \end{matrix}, \quad \begin{matrix} 2_1 2_2 \\ I \end{matrix}, \quad \begin{matrix} 2_1 \\ I \\ 2_2 \end{matrix}, \quad \begin{matrix} 2_2 \\ I \\ 2_1 \end{matrix}$$

Much easier to handle is

<u>Example 2</u>. $G = SL(2,5)$ in characteristic 5. Here we just state the decomposition matrix. Again a character or a module (or a Brauer character) is identified by its dimension and, if necessary, an index

(6)

	I	3	2	4	5
1	1	0	0	0	0
3_1	0	1	0	0	0
3_2	0	1	0	0	0
4	1	1	0	0	0
2_1	0	0	1	0	0
2_2	0	0	1	0	0
6	0	0	1	1	0
5	0	0	0	0	1

The p.i.m.'s have the following Loewy structure

(7)

P_I	P_3	P_2	P_4	P_5
I	3	2	4	5
3	I 3	4 2	2	
I	3	2	4	

as the reader may easily check. Note that

(8) $$\mathrm{Ext}^1_{F[G]}(2,2) \simeq \mathrm{Ext}^1_{F[G]}(3,3) \simeq F.$$

Example 3. Let us compare the p.i.m.'s of $Q = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, of $H = Q.K$, the Frobenius group with kernel Q and complement $K \simeq \mathbb{Z}_7$, and of $G = Q.M \simeq H.L$, semidirect product of Q and the Frobenius group of order 21, which is a subgroup of $\mathrm{Aut}(Q)$. Thus G contains a normal subgroup of index 3 isomorphic to H.

Let P be a p.i.m. of G. By Corollary 9.6, the Loewy series of P, $P_{\downarrow H}$ and $P_{\downarrow Q}$ coincide. Let F be an algebraically closed field of characteristic 2. As usual, the projective cover of a module E is denoted by P_E, and for $T \leq G$ the trivial $F[T]$-module is denoted by I_T.

As pointed out in Example 10.1, the Loewy series of P_{I_Q} is

(9)

$$\begin{matrix} & I & \\ I & I & I \\ I & I & I \\ & I & \end{matrix}$$

Let 1 denote a non-trivial simple $F[H]$-module. Then 1 is a 1-dimensional representation of $F[\mathbb{Z}_7]$. Set $i = 1^{(\otimes i)} = (1^{(\otimes i-1)})\otimes_F 1$. Then 1, 2 and 4 are algebraic conjugate, and so are 3, 5 and 6. Moreover, $1^* = 6$. The reader will probably agree that the decomposition matrix has the form

(10)

	I_H	1	2	3	4	5	6
1	1						
1_1		1					
1_2			1				
1_3				1			
1_4					1		
1_5						1	
1_6							1
7	1	1	1	1	1	1	1

which shows that F[H] has only one block. Moreover, we observe that all simples are liftable. Hence, the trivial module $I_H = 7$ must extend some other module (no subgroup of index 2), say 1. Obviously, P_{I_H} is algebraically invariant, whence its Loewy layers are, too. It immediately follows from (9) that the Loewy layers of the p.i.m.'s are

(11)

P_{I_H}	$P_1 = P_{I_H} \otimes 1$
I_H	1
1 2 4	2 5 3
3 5 6	I_H 6 4
I_H	1

etc.

If we go to F[G], the simples again are just the simples of F[M], as $Q \trianglelefteq G$. Thus the simples consist of 3 1-dimensionals, I_H, 1, 1^*, and 2 3-dimensionals 3, 3^*. The decomposition matrix is

(12)

	I	1	1^*	3	3^*
1	1	0	0	0	0
1_1	0	1	0	0	0
1_2	0	0	1	0	0
3	0	0	0	1	0
3^*	0	0	0	0	1
7	1	0	0	1	1
7_1	0	1	0	1	1
7_2	0	0	1	1	1

Using Corollary 9.6 again, it easily follows that P_{I_G}, P_1 and P_{1^*} restricted to $F[H]$ all equal P_{I_H}, as $1_{\downarrow H} = 1^*_{\downarrow H} = I_H$. Moreover, we may as well choose our notation so, that $3_{\downarrow H} = 1 \oplus 2 \oplus 4$, which shows that $P_{3\downarrow F[H]} = P_1 \oplus P_2 \oplus P_4$.

With all this information, it now follows that the p.i.m.'s of $F[G]$ have the following Loewy series:

(13)

P_{I_G}	P_1	P_{1^*}	P_3	P_{3^*}
I_G	1	1^*	3	3^*
3	3	3	$3\ 3^*\ 3^*$	$I_G\ 1\ 1^*\ 3\ 3^*$
3^*	3^*	3^*	$I_G\ 1\ 1^*\ 3\ 3^*$	$3^*\ 3\ 3$
I_G	1	1^*	3	3^*

Example 4. If we return to Example 1 and replace our field with GF(2), we will find that GF(2)[SL(2,4)] has 3 isomorphism classes of simple modules, I, 4_0 and 4, where $4_0 \otimes_{GF(2)} GF(4) = 2_1 \oplus 2_2$. It follows from (4) that the Loewy series of the p.i.m.'s are

	P_I	P_{4_0}	P_4
	I	4_0	4
	4_0	I I	
(14)	I I	4_0	
	4_0	I I	
	I	4_0	

and thus the Cartan matrix is

$$
\text{(15)} \qquad \begin{Bmatrix} 4 & 4 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{Bmatrix}
$$

which is <u>not</u> symmetric.

Based on these examples (in particular Exp. 3) and numerous others, we suggest the following.

<u>Conjecture</u>: Let J denote the radical of $F[G]$, F a field. Then J^i/J^{i+1} is self-dual for all i.

We will see in Lemma II.1.5 that this is equivalent to

<u>Conjecture</u>: Let P be a projective $F[G]$-module. Then $\dim_F(PJ^i/PJ^{i+1}) = \dim_F(P^*J^i/P^*J^{i+1})$ for all i

CHAPTER II. INDECOMPOSABLE MODULES AND RELATIVE PROJECTIVITY

1. The trace maps and the Nakayama relations.

Notation: In the following, we let Θ be a principal ideal domain and G a finite group. We denote the class of right $\Theta[G]$-modules which are free (and finitely generated) over Θ by $\mathbf{M}_\Theta(G)$.

Let $H \leq G$. If $M \in \mathbf{M}_\Theta(G)$, we denote the restriction of M to $\Theta[H]$ by $M_{\downarrow H}$ and, if $N \in \mathbf{M}_\Theta(H)$, the induction $N \otimes_{\Theta[H]} \Theta[G]$ of N to $\Theta[G]$ by $N^{\uparrow G}$, or $N\uparrow_H^G$, if it is desirable to specify which group we induce from. Thus $N^{\uparrow G} = \sum_{g_i \in H\backslash G} N \otimes g_i$, where $H\backslash G$ denotes an arbitrary right transversal of H in G.

If $A, B \in \mathbf{M}_\Theta(G)$, the set of fixed points of G in A is denoted by A^G. Also, we set

$$(A, B) = \operatorname{Hom}_\Theta(A, B) \quad . \tag{1}$$

Then (A, B) is a finitely generated free Θ-module, too. Moreover, we observe that (A, B) in fact is an $\Theta[G]$-module, if we for $\phi \in (A, B)$ and $g \in G$ define

$$[\phi g](a) = \phi(ag^{-1})g \tag{2}$$

for all $a \in A$. In particular, as we saw in Chapter I, Section 6, the dual of A, which we denote by A^*, is (A, Θ), where G acts trivially on $\mathcal{O}$.

Finally, we observe that with this notation

(3) $$(A, B)^G = \mathrm{Hom}_{\mathcal{O}[G]}(A, B) \quad .$$

There is no doubt that the most important basic tool in representation theory of finite groups is restriction to and induction from subgroups. Therefore the following definition plays a very central role.

Definition 1.1. Let $N \in \mathbf{M}_{\mathcal{O}}(G)$. Define an $\mathcal{O}$-linear map

(4) $$\mathrm{Tr}_H^G : N^H \longrightarrow (N^{\uparrow G})^G$$

by

(5) $$\mathrm{Tr}_H^G(a) = \sum_{g_i \in H \setminus G} a \otimes g_i$$

where $H \setminus G$ denotes an arbitrary right transversal of H in G. We leave it to the reader to check that $\mathrm{Tr}_H^G(a)$ is independent of the choice of $H \setminus G$ and that $\mathrm{Tr}_H^G(a)$ indeed belongs to $(N^{\uparrow G})^G$ if $a \in N^H$. This map is called the (exterior) trace map.

If furthermore $N \in \mathbf{M}_{\mathcal{O}}(G)$, we define the (interior) trace map

(6) $$\overline{\mathrm{Tr}}_H^G : N^H \longrightarrow N^G$$

by

(7) $$\overline{\mathrm{Tr}}_H^G(a) = \sum_{g_i \in H \setminus G} a g_i \quad .$$

Thus $\overline{\mathrm{Tr}}_H^G = \varepsilon \circ \mathrm{Tr}_H^G$, where $\varepsilon : N^{\uparrow G} \longrightarrow N$ is the map $\sum_i a_i \otimes g_i \longrightarrow \sum_i a_i g_i$.

As soon as we have got accustomed to the trace map, we will use the notation Tr for both of them, unless both occur at the same time.

This is the most general form of the trace map. Of particular interest will be the case where the trace map is applied to a module of the form (X, Y), where $X \in \mathbf{M}_{\Theta}(G)$ and $Y \in \mathbf{M}_{\Theta}(H)$, which has a natural structure as an $\Theta[H]$-module, as we have just seen. Thus

$$(8) \qquad \mathrm{Tr}_H^G : (X, Y)^H \longrightarrow ((X, Y)\uparrow_H^G)^G \quad .$$

We now observe however, that

Lemma 1.2. As $\Theta[G]$-modules,

$$(9) \qquad (X, Y)\uparrow_H^G \simeq (X, Y\uparrow_H^G)$$

and the isomorphism is given by

$$(10) \qquad \sum_{g_i \in H \backslash G} \phi_i \otimes g_i \longrightarrow \psi : x \longrightarrow \sum_i \phi_i(xg_i^{-1}) \otimes g_i \quad .$$

Likewise,

$$(11) \qquad (X, Y)\uparrow_H^G \simeq (X\uparrow_H^G, Y)$$

as $\Theta[G]$-modules, and here the isomorphism is

$$(12) \qquad \sum_{g_i \in H \backslash G} \phi_i \otimes g_i \longrightarrow \psi : \sum_i x_i \otimes g_i \longrightarrow \sum_i \phi_i (x_i) g_i$$

Proof: Obviously, the map in (10) is Θ-linear and bijective. It remains to check G-linearity. Let $g \in G$, and let $\{h_j\}_j$ be determined by $g_i g = h_j g_j$ for all i. Then $(\sum_i \phi_i \otimes g_i)g = \sum \phi_i h_j \otimes g_j$ is mapped to ψ_g:

$$(13) \qquad x \longrightarrow \sum_i [\phi_i h_j](xg_j^{-1}) \otimes g_j$$
$$= \sum_i \phi_i (xg_j^{-1}h_j^{-1})h_j \otimes g_j$$
$$= \sum_i \phi_i(xg_j^{-1}h_j^{-1}) \otimes h_j g_j$$
$$= \sum_i \phi_i(xg^{-1}g_i^{-1}) \otimes g_i g = [\psi g](x) \quad .$$

The second isomorphism is handled in a similar way. Alternatively, it follows from the fi st by using the fact that if $A, B \in \mathbf{M}_\Theta(G)$, then $(A, B)^* \simeq (B, A)$ as $\Theta[G]$-modules (see Section 6).

Remark: We observe that Theorem I.6.4 is just a special case of this result.

It now follows that (8) induces a trace map (which we will denote by Tr_H^G, too):

$$(14) \qquad Tr_H^G : (X, Y)^H \longrightarrow (X, Y\uparrow_H^G)^G$$

given by

$$(15) \qquad [Tr_H^G(\phi)](x) = \sum_{g_i \in H \backslash G} \phi(xg_i^{-1}) \otimes g_i$$

and likewise, if moreover $Y \in \mathbf{M}_\Theta(G)$,

$$(16) \qquad \bar{T}r_H^G : (X, Y)^H \longrightarrow (X, Y)^G$$

by

$$(17) \qquad [\overline{\mathrm{Tr}}_H^G(\phi)](x) = \sum_{g_i \in H\backslash G} [\phi g_i](x) = \sum_i \phi(xg_i^{-1})g_i$$

which fortunately is identical to the trace map of (7). Also, $\overline{\mathrm{Tr}}_H^G = \hat{\varepsilon} \circ \mathrm{Tr}_H^G$ in this setup, where $\hat{\varepsilon} : (X, Y{\uparrow}_H^G) \longrightarrow (X, Y)$ is induced from the canonical map ε defined in Definition 1.1.

Similarly, we get a dual trace map

$$(18) \qquad \mathrm{Fr}_H^G : (Y, X)^H \longrightarrow (Y{\uparrow}_H^G, X)^G$$

given by

$$(19) \qquad [\mathrm{Fr}_H^G(\phi)](\sum_{g_i \in H\backslash G} y_i \otimes g_i) = \sum_i \phi(y_i)g_i \quad .$$

We observe that $\mathrm{Fr}_H^G(\phi) = \mathrm{Tr}_H^G(\phi^*)^*$, where $\phi^* \in (X^*, Y^*)$ is the dual map induced from $\phi \in (X, Y)$ (see Ch. I, Sec. 6, (10)).

In particular, any result we prove about the general trace map in (4) translates in a natural way into similar results for the special and dual trace maps in (14) and (18). One advantage of this is that in order to develop block theory, we need the trace map defined in (4), while in the theory of modules we often need the special and dual trace maps, rather. Another advantage is that the general trace map is so straightforward to deal with. Let us demonstrate this.

Theorem 1.3. With the notation of Definition 1.1,

$$(20) \qquad \mathrm{Tr}_H^G : N^H \longrightarrow (N^{\uparrow G})^G$$

is an Θ-isomorphism.

Proof: By definition, Tr_H^G is injective. On the other hand, if $\sum_i a_i \otimes g_i \in (N\uparrow^G)^G$, then $\sum_i a_i \otimes g_i g = \sum_i a_i \otimes g_i$ for all $g \in G$, and consequently $a := a_i = a_j$ for all i, j. Thus $\sum_i a_i \otimes g_i = \mathrm{Tr}_H^G(a)$.

Corollary 1.4 (The Nakayama Relations).

i) Let $X \in \mathbf{M}_\Theta(G)$, $Y \in \mathbf{M}_\Theta(H)$. Then

$$\mathrm{Tr}_H^G : (X, Y)^H \longrightarrow (X, Y\uparrow_H^G)^G \tag{21}$$

is an Θ-isomorphism. Its inverse is the map induced from $\mathrm{Pr}_H^G : Y\uparrow_H^G \longrightarrow Y$, mapping $\sum y_i \otimes g_i$ to $y_1 g_1$, where $g_1 \in H$.

ii) Let $X \in \mathbf{M}_\Theta(H)$, $Y \in \mathbf{M}_\Theta(G)$. Then

$$\mathrm{Fr}_H^G : (X, Y)^H \longrightarrow (X\uparrow_H^G, Y)^G \tag{22}$$

is an Θ-isomorphism. Its inverse is the map induced from $\mathrm{In}_H^G : X \longrightarrow X\uparrow_H^G$, mapping x to $xg_1^{-1} \otimes g_1$, where $g_1 \in H$.

In particular, if $X, Y \in \mathbf{M}_\Theta(G)$, then $(X, Y\uparrow_H^G) \simeq (X\uparrow_H^G, Y)$ as an $\Theta[G]$-module, and we have the following diagram

$$\begin{array}{ccccc} & & (X, Y\uparrow_H^G)^G & & \\ & \mathrm{Tr}_H^G \nearrow \swarrow \hat{\mathrm{Pr}}_H^G & & \searrow \hat{\varepsilon} & \\ (X, Y) & & \longrightarrow & & (X, Y)^G \\ & \mathrm{Fr}_H^G \searrow \nwarrow \hat{\mathrm{In}}_H^G & & & \\ & & (X\uparrow_H^G, Y)^G & & \end{array} \tag{23}$$

Proof: By our discussion above.

Notice however, that conversely, Theorem 1.3 is a special case of The Nakayama Relations. Indeed, if I denotes Θ with trivial G-action, and $M \in \mathbf{M}_\Theta(H)$, (21) states that $(I, Y)^H \simeq (I, Y\uparrow_H^G)^G$, which is just another way of expressing (20).

Example 1. Let G be an arbitrary group, let F be any field of characteristic p, and let $Q \leq G$ be a p-group. Let E be any simple $F[G]$-module, and denote the trivial $F[Q]$-module by I_Q. Then $I_Q^{\uparrow G}$ contains a submodule and a factor module isomorphic to E. This follows from (21) and (22), as any composition factor of $E_{\downarrow Q}$ is isomorphic to I_Q.

Example 2. Let Q be a p-group. For $V \leq Q$, denote the trivial $F[V]$-module by I_V. Then $I_V^{\uparrow Q}$ is indecomposable. Indeed, by Theorem 1.3,

$$(24) \qquad \dim_F(\mathrm{Soc}(I_V^{\uparrow Q})) = \dim_F(\mathrm{Soc}(I_V)) = 1$$

as Q is a p-group. Consequently, $I_V^{\uparrow Q}$ cannot be the direct sum of two modules.

Likewise, $\hat{I}_V^{\uparrow Q}$ is indecomposable, if $\hat{I}_V$ denotes the trivial $R[V]$-module.

The following application will be of great use later.

Lemma 1.5. Let F be a field of characteristic p and let E be a simple $F[G]$-module. Let P denote the projective cover of E and J the radical of $F[G]$. Then

$$(25) \qquad \dim(PJ^i/PJ^{i+1}) = \dim((J^i/J^{i+1}, E^*)^G)$$

for all i.

Proof: Denote the ith socle of $F[G]$ by S_i and recall that $S_i^* \cong F[G]/J^i$, as $F[G]$ is self-dual. Now, if V is a vector space over F, denote the dimension of V by $d(V)$. Then

$$(26) \qquad d(PJ^i/PJ^{i+1}) = d(P/PJ^{i+1}) - d(P/PJ^i)$$
$$= d((P/PJ^{i+1}, F)) - d((P/PJ^i, F))$$
$$= d((P/PJ^{i+1}, F[G])^G) - d((P/PJ^i, F[G])^G)$$

by the Nakayama relations

$$= d((P, S_{i+1})^G) - d((P, S_i)^G)$$
$$= d((P, S_{i+1}/S_i)^G)$$

as P is projective

$$= d((J^i/J^{i+1}, P^*)^G)$$

from which (25) follows.

We proceed to reveal some of the most important properties of the general trace map, and leave in some cases the task of translating these results to the special and dual trace maps as an exercise.

Lemma 1.6. Let $K \leq L \leq G$, and let $g \in G$. Then
i) $Tr_K^G = Tr_L^G \circ Tr_K^L$ and $\bar{T}r_K^G = \bar{T}r_L^G \circ \bar{T}r_K^L$.
Let $M \in \mathbf{M}_{\Theta}(G)$. Then
ii) $\bar{T}r_K^G(M^K) \subseteq \bar{T}r_L^G(M^L)$.
iii) $\bar{T}r_H^G(M^H) = \bar{T}r_{H^g}^G(M^{H^g})$ for all $H \leq G$, $g \in G$.

Proof: By definition.

Next we recall the Mackey decomposition of a module: Let $H, K \leq G$, and let $N \in \mathbf{M}_{\Theta}(K)$. Then

(27) $$N^{\uparrow G}_{\downarrow H} \simeq \bigoplus_{g_i \in K\backslash G/H} ((N \otimes g_i)_{\downarrow K^{g_i} \cap H})^{\uparrow H}$$

as $\Theta[H]$-modules, where $K\backslash G/H$ denotes an arbitrary transversal of double (K, H)-cosets.

Similarly, we have a Mackey decomposition of the trace map.

Theorem 1.7 (Mackey decomposition). Let $H, K \leq G$ and let $N \in \mathbf{M}_\Theta(K)$. Then

(28) $$\mathrm{Tr}_K^G(x) = \sum_{g_i \in K\backslash G/H} \mathrm{Tr}^H_{g_i^{-1}Kg_i \cap H}(x \otimes g_i)$$

for all $x \in N^K$.

A similar result of course holds for the interior trace map if $N \in \mathbf{M}_\Theta(G)$.

Proof: Just as the Mackey decomposition for modules, this follows from the fact that if $G = \bigcup_{g_i \in K\backslash G/H} Kg_iH$, and $Kg_iH = \bigcup_j Kb_{ij}$, disjoint union, then $\{b_{ij}\}$ is a right transversal of K in G. On the other hand if $\{h_{ij}\}$ is a right transversal of $g_i^{-1}Kg_i \cap H$ in H, then $Kg_iH = \bigcup Kg_ih_{ij}$, disjoint union.

Since it is always clear if one should use the exterior or the interior trace map, we have as mentioned earlier chosen in the following to use the notation Tr in both cases, unless both occur together.

The assumptions in the following result may seem a little artificial, but have been chosen in order to avoid repetitions.

Theorem 1.8. Let $A, B, C \in \mathbf{M}_\Theta(G)$, and let $* : A \times B \longrightarrow C$ be an Θ-linear map such that whenever $L \leq G$ and

$\phi \in A^L$, $\psi \in B^L$, then the maps $B \longrightarrow C$ and $A \longrightarrow C$ defined by $b \longrightarrow \phi * b$ and $a \longrightarrow a * \psi$ are $O[L]$-homomorphisms.

Let $H, K \leq G$ and choose $a \in A^K$, $b \in B^H$. Then

$$(29) \qquad Tr_K^G(a) * Tr_H^G(b) = \sum_{g_i \in K \backslash G/H} Tr^G_{g_i^{-1}Kg_i \cap H}(ag_i * b) \quad .$$

Proof: Our general condition yields that

$$(30) \qquad Tr_K^G(a) * Tr_H^G(b) = Tr_H^G(Tr_K^G(a) * b)$$

which by ii) equals

$$(31) \qquad Tr_H^G\Big(\sum_{g_i \in K \backslash G/H} Tr^H_{g_i^{-1}Kg_i \cap H}(ag_i) * b\Big)$$

$$= Tr_H^G\Big(\sum_{g_i \in K \backslash G/H} Tr^H_{g_i^{-1}Kg_i \cap H}(ag_i * b)\Big)$$

again because of our convenient assumption.

Notation: Let $H \leq G$, and let $M \in \mathbf{M}_\Theta(G)$. We then set $M_H^G = Tr_H^G(M^H)$. More generally, if $\mathcal{H}$ is any family of subgroups of G, we set

$$(32) \qquad M_{\mathcal{H}}^G = \sum_{U \in \mathcal{H}} M_U^G$$

and $M^{\mathcal{H},G} = M^G/M_{\mathcal{H}}^G$.

As the first application of Mackey decomposition, we now state

Corollary 1.9. Let $M \in \mathbf{M}_\Theta(G)$, and let $V, H \leq G$. Set

$$(33) \qquad \mathfrak{X} = \{V^\gamma \cap V \mid \gamma \notin H\}, \quad \mathcal{Y} = \{V^\gamma \cap H \mid \gamma \notin H\} \quad .$$

Then

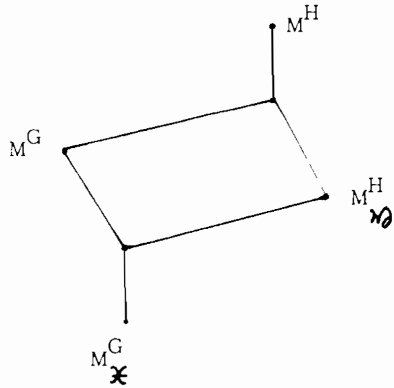
i) $Tr_V^G(m) = Tr_V^H(m) \bmod M^H_{\mathcal{Y}}$ for $m \in M^V$. In particular,

ii) $M_V^H \subseteq M_V^G + M^H_{\mathcal{Y}}$

iii) $M^G_{\mathcal{X}} \subseteq M^H_{\mathcal{Y}}$.

Thus we have the following diagram

(34)



Proof: i) follows directly from Mackey decomposition, and ii) follows from i). Finally, if $W \in \mathcal{X}$ and $g \in G$ arbitrary, $W^g \cap H$ is a subgroup of some element of $\mathcal{Y}$. Thus Mackey decomposition implies iii) as well.

Corollary 1.10. Let $A \in \mathbf{M}_\Theta(G)$ be a ring. Assume furthermore the assumption of Theorem 1.8 is satisfied for $A = B = C$ and the multiplicative structure of A. Then

i) $A^G_{\mathcal{H}}$ is an ideal in A^G for any family $\mathcal{H}$ of subgroups of G.

ii) With the notation of Corollary 1.9,

(35) $$Tr_H^G(a)Tr_H^G(b) = Tr_H^G(ab) \bmod A^G_{\mathcal{X}}$$

for all $a, b \in A_V^H$.

Proof: i) follows directly from the definition, actually.

ii) Set $a = Tr_V^H(\alpha)$, $b = Tr_V^H(\beta)$, where $\alpha, \beta \in A^V$. Then

$$(36) \qquad Tr_H^G(a)Tr_H^G(b) = Tr_V^G(\alpha)Tr_V^G(\beta)$$

by (29),

$$= \sum_{g_i \in V\backslash G/V} Tr^G_{g_i^{-1}Vg_i \cap V}(\alpha g_i \beta)$$

$$= \sum_{g_i \in V\backslash H/V} Tr^G_{g_i^{-1}Vg_i \cap V}(\alpha g_i \beta) \bmod A^G_{\mathfrak{X}}$$

$$= Tr_H^G\Big(\sum_{g_i \in V\backslash H/V} Tr^H_{g_i^{-1}Vg_i \cap V}(\alpha g_i \beta)\Big) \bmod A^G_{\mathfrak{X}}$$

$$= Tr_H^G(ab) \bmod A^G_{\mathfrak{X}}$$

again by (29).

Example 3: If $M \in \mathbf{M}_{\Theta}(G)$, then $A = (M, M)^G$ satisfies the assumption of Theorem 1.7, and thus $A^G_{\mathfrak{X}}$ is an ideal in A. Hence Tr_H^G induces a ring homomorphism on $A^{\mathfrak{X},G}$, a fact we will take great advantage of later. But at this stage it seems natural first to investigate how one ensures that $A^G_{\mathfrak{X}}$ is a proper ideal in A^G. This will be analyzed in the following sections.

2. Relative projectivity.

Before we continue, there are a few points we have to make in view of Ch. I, 14. Namely, the concepts of projective cover and injective hull may be extended from the original in Ch. I, 10. Let (F, R, S) be a p-modular system, and let $M \in \mathbf{M}_R(G)$. In view of Lemma I.14.4, we may then define the projective cover $P_M \in \mathbf{M}_R(G)$ of M as the lift of the projective $F[G]$-cover of $M/M\pi$. We then define

ΩM by

(1) $$0 \longrightarrow \Omega M \longrightarrow P_M \longrightarrow M \longrightarrow 0$$

and observe that ΩM is an R-pure submodule of P_M, as M is R-free. It follows that $\Omega M/\Omega M\pi \simeq \Omega(M/M\pi)$. Likewise, we may define the (R-)injective hull I_M of M (cf. Theorem 1.14.10 and its proof) and define $\mho M$ by

(2) $$0 \longrightarrow M \longrightarrow I_M \longrightarrow \mho M \longrightarrow 0 \quad .$$

Again, M is an R-pure submodule of I_M, and thus $\mho M$ is R-free.

It immediately follows, that Lemma I.10.2 through I.10.6 holds for elements in $\mathbf{M}_R(G)$, and in particular, we will talk about Schanuel's lemma for modules in this category.

We now introduce the concept of relative projectivity.

Let Θ be a principal ideal domain such that Krull-Schmidt holds for modules in $\mathbf{M}_\Theta(H)$ for all $H \leq G$. If $M \in \mathbf{M}_\Theta(G)$ and X is isomorphic to a direct summand of M, we write $X|M$.

Definition 2.1. Let $M \in \mathbf{M}_\Theta(G)$ and let $H \leq G$. Then M is called H-projective, or projective relative to H, if there exists $N \in \mathbf{M}_\Theta(H)$ with $M|N^{\uparrow G}$.

For $A, B \in \mathbf{M}_\Theta(G)$, we recall that $(A, B)_H^G = \mathrm{Tr}_H^G((A, B)^H) \subseteq (A, B)^G$. This is the subspace of H-projective homomorphisms.

Similarly, if $\mathcal{H}$ is a family of subgroups, M is called $\mathcal{H}$-projective if M is U-projective for some $U \in \mathcal{H}$, and $(A, B)_{\mathcal{H}}^G$ is called the space of $\mathcal{H}$-projective homomorphisms.

Lemma 2.2. Let $M \in \mathbf{M}_\Theta(G)$, and let $H \leq G$. Then

i) M is 1-projective if and only if M is projective.

ii) Let M be H-projective. Then M is H^g-projective as well for all $g \in G$.

iii) If $M | N^{\uparrow G}$ for some $N \in \mathbf{M}_{\Theta}(H)$ and N is K-projective for some $K \leq H$, then M is K-projective as well.

iv) Let $K \underset{G}{\leq} H$. If M is K-projective, then M is H-projective.

v) M is H-projective if and only if M^* is H-projective.

vi) Assume Θ is a p-adic ring or a field. Then M is H-projective if and only if ΩM is H-projective.

Proof: i) - v) are easy exercises.

vi) is a straightforward exercise, if one remembers that Ω is additive and uses Lemma 1.10.3 & 5.

Let $A, B \in \mathbf{M}_{\Theta}(G)$, and let $H \leq G$. Recall $(A, B)_H^G = (A, B)_{H^g}^G$ for all $g \in G$, and that $(A, B)_H^G \subseteq (A, B)_K^G$ if $H \underset{G}{\leq} K$, as we saw in Lemma 1.5.

We now have the following connection between relatively projective modules and relatively projective homomorphisms.

The next result is well-known and appears in Green (1974) in the special case where $H = 1$.

Theorem 2.3. With the notation above, let $\alpha \in (A, B)^G$. Then the following are equivalent:

i) $\alpha \in (A, B)_H^G$

ii) α factors through an H-projective module. In other words, there exists an H-projective module $M \in \mathbf{M}_{\Theta}(G)$ and homomorphisms ψ_1, ψ_2 such that

(3)
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & {\scriptstyle\psi_1}\searrow \quad \nearrow{\scriptstyle\psi_2} & \\ & M & \end{array}$$

commutes.

iii) α factors through $(B_{\downarrow H})^{\uparrow G}$.

Proof: That i) implies iii) follows from 1, (7). Indeed, if $M = (B_{\downarrow H})^{\uparrow G} = \sum_{g_i \in H \setminus G} B_{\downarrow H} \otimes g_i$ and $\alpha = Tr_H^G(\gamma)$ for $\gamma \in (A, B)^H$, then $\psi_1 := Tr_H^G(\gamma) \in (A, M)^G$, and if we define $\psi_2 = \varepsilon$, where as earlier $\varepsilon \in (M, B)^G$ is defined by $\varepsilon(\sum x_i \otimes g_i) = \sum x_i g_i$, we see that (3) holds.

Obviously, iii) implies ii).

Finally, given (3), where M is H-projective, let $N \in \mathbf{M}_{\Theta}(H)$ with $M | N^{\uparrow G}$. Then (3) may be replaced by

(4)
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & {\scriptstyle\psi_1}\searrow \quad \nearrow{\scriptstyle\psi_2} & \\ & N^{\uparrow G} & \end{array}$$

By the Nakayama relations, let $\phi_1 \in (A, N)^H$ with $Tr_H^G(\phi_1) = \psi_1$ and $\phi_2 \in (N, B)^H$ with $Fr_H^G(\phi_2) = \psi_2$. Then, for any $a \in A$,

(5)
$$\begin{aligned} \alpha(a) = \psi_2 \circ \psi_1(a) &= Fr_H^G(\phi_2) \circ Tr_H^G(\phi_1)(a) \\ &= [Fr_H^G(\phi_2)]\Big(\sum_{g_i \in H \setminus G} \phi_1(ag_i^{-1}) \otimes g_i\Big) \\ &= \sum_{g_i \in H \setminus G} (\phi_2 \circ \phi_1(ag_i^{-1}))g_i \\ &= [Tr_H^G(\phi_2 \circ \phi_1)](a) \quad . \end{aligned}$$

We may now characterize H-projectivity in a number of ways:

Let $M \in \mathbf{M}_{\Theta}(G)$ be arbitrary. Then we always have the surjective homomorphism $\varepsilon : M^{\uparrow G} \longrightarrow M$ defined by

(6) $$\varepsilon\Big(\sum_{g_i \in H\backslash G} m_i \otimes g_i\Big) = \sum m_i g_i \quad .$$

Likewise, there is always an injective homomorphism $\nu : M \longrightarrow M^{\uparrow G}$ defined by

(7) $$\nu(m) = \sum_{g_i \in H\backslash G} m g_i^{-1} \otimes g_i \quad .$$

Corollary 2.4. With the notation above, the following are equivalent:

i) M is H-projective

ii) $M \mid (M_{\downarrow H})^{\uparrow G}$

iii) ε splits

iv) ν splits

v) Any short exact sequence

(8) $$0 \to X \to Y \to M \to 0 \quad \text{resp.} \quad 0 \to M \to X \to Y \to 0$$

splits if and only if its restriction to H splits

vi) $(M, M)^G = (M, M)_H^G$

vii) There exists $\gamma \in (M, M)^H$ such that $Tr_H^G(\gamma)$ is the identity map of M.

Proof: First of all iii) and iv) are equivalent through duality, as $\nu : M^* \longrightarrow (M^*)^{\uparrow G}$ is the same as the induced map $\varepsilon^* : M^* \longrightarrow (M^{\uparrow G})^*$ followed by the isomorphism of 1, (11).

Also, vi) is equivalent to vii) as $(M, M)_H^G$ is an ideal in

$(M, M)^G$.

Obviously, iii) implies ii), and ii) implies i) by definition. Next, if M is H-projective, the identity map of M certainly factors through an H-projective module, and thus vii) holds by Theorem 2.3. Next, vii) implies that $\varepsilon \circ \mathrm{Tr}_H^G(\gamma)$ is the identity of M as well, and thus the exact sequence

$$(10) \qquad 0 \longrightarrow \mathrm{Ker}\ \varepsilon \longrightarrow (M_{\downarrow H})^{\uparrow G} \longrightarrow M \longrightarrow 0$$

splits. Hence vii) implies iii). Similarly, if vii) holds and the first sequence of (8) has the property that its restriction to H splits (by duality we only have to deal with one of them), then γ of vii) may be factored through $Y_{\downarrow H}$, and thus the identity of M may be factored through Y by vii) and v) holds. Finally, v) yields that (10) splits, and our tour is complete.

Remark: vi) above is usually known as D. G. Higman's Criterion for projectivity (Higman (1954)). Parts of the corollary go back to this paper and Gaschütz (1952).

Corollary 2.5. Let (F, R, S) be a p-modular system, and let $\mathcal{O}$ be F or R.

Let $Q \in \mathrm{Syl}_p(G)$, and let $M \in \mathbf{M}_{\mathcal{O}}(G)$. Then M is Q-projective.

Proof: We simply observe that $\frac{1}{|G:Q|}\mathrm{Id}_M$ is well defined in $(M, M)^G$ and use vi) of Corollary 2.4.

Corollary 2.6. Let $M, N \in \mathbf{M}_F(G)$, F as above. Then

$$(11) \qquad \mathrm{Ext}^1_{F[G]}(M, N) \simeq (\Omega M, N)^{1,G} .$$

Proof: Let $\phi_1 : \Omega M \longrightarrow P_M$ be the embedding, where P_M is the projective cover of M, and Ω is the Heller operator. By definition (Ch. I, Sec. 10),

$$\text{(12)} \qquad \mathrm{Ext}^1_{F[G]}(M, N) = (\Omega M, N)^G / \hat{\phi}_1(P_M, N)^G$$

where $\hat{\phi}_1(\psi) = \psi \circ \phi_1$. Thus $\psi \circ \phi_1 \in (\Omega M, N)^G$ factors through P_M, whence is projective by Theorem 2.3. Conversely, that any projective map in fact factors through P_M follows from the following exercise in homological algebra:

Lemma 2.7. Same notation as in Corollary 2.5. Let $A, B \in \mathbf{M}_{\mathcal{E}}(G)$, and let $\alpha \in (A, B)^G$. Then the following are equivalent:

i) $\alpha \in (A, B)^G_1$.

ii) α may be factored through the injective hull of A.

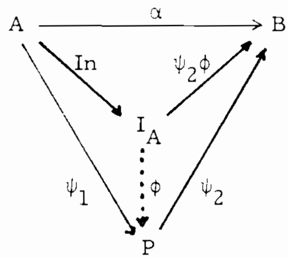
iii) α may be factored through the projective cover of B.

Proof: i) implies ii): Let $\alpha \in (A, B)^G_1$. By Theorem 2.3, there exists a projective module P such that

$$\text{(13)} \qquad \begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & B \\ & {\scriptstyle \psi_1} \searrow \quad \nearrow {\scriptstyle \psi_2} & \\ & P & \end{array}$$

commutes. Let I_A denote the ($\mathcal{E}$-) injective hull of A, and $\mathrm{In} : A \longrightarrow I_A$ the embedding. As P is injective, there exists $\phi : I_A \longrightarrow P$ such that $\psi_1 = \phi \circ \mathrm{In}$

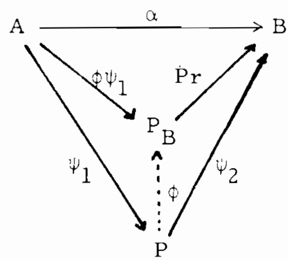
(14)



and thus $\alpha = (\psi_2 \circ \phi) \circ \mathrm{In}$ factors through I_A.

i) implies iii): Let P_B denote the projective cover of B, and $\mathrm{Pr} : P_B \longrightarrow B$ the surjection. As P is projective, there exists $\phi : P \longrightarrow P_B$ such that $\psi_2 = \mathrm{Pr} \circ \phi$. Hence $\alpha = \mathrm{Pr} \circ (\phi \circ \psi_1)$ factors

(15)



through P_B.

Finally, ii) and iii) obviously both imply i), as an injective $\mathcal{O}[G]$-module is projective.

For later use, we also mention

<u>Corollary 2.8</u>. Let $A, B \in \mathbf{M}_F(G)$, F as above, be projective-free, and let J denote the radical of $F[G]$. Let $\phi \in (A, B)_1^G$. Then $\mathrm{Soc}(A) \subseteq \mathrm{Ker}\ \phi$ and $\phi(A) \subseteq BJ$.

In particular $(A, B)_1^G = 0$ if A or B is semisimple.

Proof: By additivity, it suffices to consider the case where A and B are indecomposable and also to prove that $Soc(A) \subseteq Ker \phi$ since then duality and Theorem 2.3 yield that $\phi(A) \subseteq BJ$ is satisfied too. Now using the notation of Lemma 2.7, ϕ may be factored through I_A. Moreover, if some simple component E of Soc(A) is not in the kernel of ϕ, some direct summand of I_A isomorphic to the injective hull I_E of E intersects $Ker(\psi_2\phi)$ trivially. Thus I_E is a direct summand of B, a contradiction.

Corollary 2.9. Same notation as in Corollary 2.5. Let $A, B \in \mathbf{M}_R(G)$ and set $\bar{A} = A/A\pi$, $\bar{B} = B/B\pi$. Then

$$(16) \qquad (\bar{A}, \bar{B})_1^G = ((A, B)_1^G + (A, B)^G\pi)/(A, B)^G\pi$$

i.e., a projective map from a liftable module into a liftable module is liftable.

In particular, if S is a splitting field of S[G] and χ_A resp. χ_B is the character of $A \otimes_R S$ resp. $B \otimes_R S$, then

$$(17) \qquad \dim_F(\bar{A}, \bar{B})_1^G \leq (\chi_A, \chi_B)_G$$

and

$$(18) \qquad \dim_F(\bar{A}, \bar{A})_1^G \leq (\chi_A, \chi_A) - 1$$

if A is not projective. Thus $(M, M)_1^G = 0$ for any non-projective F[G]-module M, which lifts to a module with an irreducible character.

Proof: Let $\bar{\alpha} \in (\bar{A}, \bar{B})_1^G$, and let P be a projective R[G]-module such that $\bar{\alpha}$ factors through $\bar{\psi}_1 \in (\bar{A}, \bar{P})^G$ and $\bar{\psi}_2 \in (\bar{P}, \bar{B})^G$, where $\bar{P} = P/P\pi$. Then $\bar{\psi}_1$ lifts to $\psi_1 \in (A, P)^G$ and

$\overline{\psi}_2$ to $\psi_2 \in (P, B)^G$ by Theorem I.14.7. Thus $\overline{\alpha}$ lifts to $\alpha = \psi_2 \circ \psi_1 \in (A, B)_1^G$, and (16) follows. Now (17) follows from Lemma I.14.5.

Moreover, if $A = B$ is not projective, the identity of A does not belong to $(A, A)_1^G$, and thus $\dim_F(\overline{A}, \overline{A})_1^G < (\chi_A, \chi_A)$ by (16).

Remark: As an illustration of this remarkable result, see Example 2 of Section 13.

We end this section with the following warning if one deals with R-modules rather than F-modules.

Lemma 2.10. Same notation as above. Let $A, B \in \mathbf{M}_R(G)$, and let H be any subgroup of G. Then $(A, B)^{H,G}$ is a torsion module, or in other words, $\operatorname{rank}_R((A, B)^G) = \operatorname{rank}_R((A, B)^{H,G})$.

Proof: It suffices to consider the case $H = 1$ by Lemma 1.6. Choose n such that $|G| = u\pi^n$, where u is a unit in R. Then for any $\phi \in (A, B)^G$,

$$(19) \qquad \sum_{g \in G} (u^{-1}\phi)g = u^{-1}|G|\phi = \pi^n \phi$$

which therefore is an element of $(A, B)_1^G$. Thus $(A, B)^G \pi^n \subseteq (A, B)_1^G$, and we are done.

We end this section with mentioning the following remarkable result without proof:

Theorem 2.11 (Green (1959b). Same notation as above. Let E be a simple $F[G]$-module such that $\operatorname{Ext}^2_{F[G]}(E, E) = 0$. Then E is liftable.

Proof: Using cohomology one can lift E to $R[G]/(\pi^i)$ for all i and then take the inverse limit.

3. Vertices and sources.

We now pursue the idea of relative projectivity. Our tool is Mackey decomposition, and the basic ideas go back to Green (1959a) and (1962a).

Again we let Θ be a principal ideal domain ring such that Krull-Schmidt holds for modules in $\mathbf{M}_\Theta(H)$ for all $H \leq G$.

Observe that if $H \leq G$ and $N \in \mathbf{M}_\Theta(H)$, then always $N \mid (N^{\uparrow G})_{\downarrow H}$.

Lemma 3.1. Let $N \in \mathbf{M}_\Theta(H)$, and assume N is V-projective for some $V \leq H$. Let $(N^{\uparrow G})_{\downarrow H} \simeq N \oplus N'$. Then any indecomposable direct summand N_i of N' is $V^\gamma \cap H$-projective for some $\gamma \notin H$.

Proof: By assumption, there exists $L \in \mathbf{M}_\Theta(V)$ such that $L^{\uparrow H} \simeq N \oplus K$ for some $K \in \mathbf{M}_\Theta(H)$. Let $(K^{\uparrow G})_{\downarrow H} \simeq K \oplus K'$. Thus

$$(1) \qquad (L^{\uparrow G})_{\downarrow H} \simeq ((N \oplus K)^{\uparrow G})_{\downarrow H} \simeq N \oplus N' \oplus K \oplus K' \quad .$$

On the other hand, by Mackey decomposition,

$$(2) \qquad (L^{\uparrow G})_{\downarrow H} = \bigoplus_{\gamma_i \in V \backslash G/H} ((L \otimes \gamma_i)_{\downarrow \gamma_i^{-1} V \gamma_i \cap H})^{\uparrow H} \quad .$$

If we choose our notation so that $\gamma_1 \in H$, then

$$(3) \qquad ((L \otimes \gamma_1)_{\downarrow V \cap H})^{\uparrow H} = (L \otimes \gamma_1)^{\uparrow H} \simeq L^{\uparrow H} \simeq N \oplus K \quad ,$$

so by Krull-Schmidt, each indecomposable direct summand of N' is isomorphic to a direct summand of some $((L \otimes \gamma_i)_{\downarrow \gamma_i^{-1} V \gamma_i \cap H})^{\uparrow H}$, where $\gamma_i \notin H$, whence is $V^{\gamma_i} \cap H$-projective.

Definition 3.2. Let $M \in \mathbf{M}_\theta(G)$ be indecomposable and set $Pr(M) = \{H \leq G \mid M$ is H-projective$\}$. The minimal elements (by order) in $Pr(M)$ are called the vertices of M.

We now have the following characterization of the vertices of M.

Theorem 3.3. Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable. Let V be a vertex of M. Then

i) Let $H \leq G$, and let $M_{\downarrow H} \simeq \bigoplus_i N_i$, where H_i is indecomposable. Let V_i be a vertex of M_i. Then $V_i \underset{G}{\leq} V$.

Assume furthermore that $H \in Pr(M)$. Then

ii) $V \underset{G}{\leq} H$. In particular, the vertices of M are uniquely determined up to conjugacy in G.

iii) Assume $V \leq H$. With the notation of i), there exists an i_0 such that $V_{i_0} \underset{H}{=} V$.

Proof: Let $L \in \mathbf{M}_\Theta(V)$ such that $M | L^{\uparrow G}$. By Mackey decomposition,

$$(4) \qquad (L^{\uparrow G})_{\downarrow H} \simeq \bigoplus_{\gamma_i \in V \backslash G / H} ((L \otimes \gamma_i)_{\downarrow \gamma_i^{-1} V \gamma_i \cap H})^{\uparrow H} .$$

As $M_{\downarrow H} | (L^{\uparrow G})_{\downarrow H}$, (4) yields that any indecomposable direct summand N_i of $M_{\downarrow H}$ is a direct summand of a module induced from $\gamma_j^{-1} V \gamma_j \cap H$ for some j. Hence i) follows from ii).

If furthermore M is H-projective, $M | (M_{\downarrow H})^{\uparrow G}$ and thus $M | N_i^{\uparrow G}$ for some i. Hence $\gamma_j^{-1} V \gamma_j \cap H \in Pr(M)$ by what we saw above. Now minimality of V forces $V^{\gamma_i} \cap H = V^{\gamma_i}$, and ii) follows.

Finally, as $M | (M_{\downarrow V})^{\uparrow G}$, there exists an indecomposable direct summand U of $M_{\downarrow V}$ with vertex V, by choice of V and Lemma 2.2 iii), such that $M | U^{\uparrow G}$. In particular, if $V \leq H$, there exists an i_0 such that $U | N_{i_0 \downarrow V}$. But then $V \underset{H}{\leq} V_{i_0}$ by ii), and thus $V \underset{H}{=} V_{i_0}$ in fact, by i).

Remark: This result will be improved later. Indeed, it turns out that i_0 may be chosen so in iii) that moreover $M | N_{i_0}^{\uparrow G}$ (Burry (1979)).

This result now allows us to add the following supplement to Lemma 3.1.

Lemma 3.4. Let $N \in \mathbf{M}_{\Theta}(H)$ and assume N is V-projective for some $V \leq H$. Let $N^{\uparrow G} = \bigoplus_i M_i$ with M_i indecomposable. Then there exists an i_0 such that

i) $N | M_{i_0 \downarrow H}$

ii) M_i is $V^{\gamma_i} \cap V$-projective for some $\gamma_i \notin H$, for all $i \neq i_0$.

Proof: Let $W_i \leq V$ be a vertex of M_i, in view of Lemma 3.1 and Theorem 3.3. Furthermore, Lemma 3.1 asserts the existence of an i_0 such that $N | M_{i_0 \downarrow H}$, while any component of $M_{i \downarrow H}$, $i \neq i_0$ is $V^{\gamma_i} \cap H$-projective for some $\gamma_j \in H$. But then there exists a j_0 and $h_{j_0} \in H$ such that $W_i^{h_{j_0}} \leq V^{\gamma_{j_0}} \cap H$ by Theorem 3.3, from which the statement follows.

Example 1. It immediately follows from Lemmas 2.2 v) and vi), that if $M \in \mathbf{M}_{\Theta}(G)$ is indecomposable with V as vertex, then M^*, ΩM (and $\mho M$) all have V as vertex as well.

Example 2. Let (F, R, S) be a p-modular system and let

Θ equal F or R. Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable. Then a vertex of M is a p-group by Corollary 2.5.

Example 3. Same notation as above. Let Q be a p-group. Then the trivial $\Theta[Q]$-module 1 has vertex Q.

Indeed by Section 1, Example 2, if $V \leq Q$, $(I_{\downarrow V})^{\uparrow Q}$ is indecomposable and thus has I as a direct summand if and only if $V = Q$.

Lemma 3.5. Same notation as above. Let M be an indecomposable $\Theta[G]$-module, and let H denote the kernel of M. Let V be a vertex (which is a p-group by Example 1). Then $V \cap H \in \mathrm{Syl}_p(H)$.

Proof: Let $Q \in \mathrm{Syl}_p(H)$. Then Q acts trivially on $M_{\downarrow Q}$, and thus any direct summand of $M_{\downarrow Q}$ has vertex Q. Hence $Q^g \leq V$ for some $g \in G$ by Theorem 3.3 and the statement follows as H is normal in G.

The next result, which provides a more conceptual proof and a significant generalization of Proposition I.15.8 suggests how useful vertices may be.

Proposition 3.6. Let S be a splitting field of S[G] and let M be an indecomposable R[G]-module. Denote the character of $M \otimes_R S$ by χ, and let V be a vertex of V.

Let $x \in G$ and assume the p-part of x (see Section 1, 13) is not conjugate to an element of V. Then $\chi(x) = 0$.

Proof: Let $\hat{R}$ denote an extension of R which contains an ord(x)'th root of unity. Then any direct summand of $M \otimes_R \hat{R}$ is obviously V-projective, and it therefore suffices to prove the statement for the case $R = \hat{R}$.

Let $H = \langle x \rangle$ and $K = \langle x^p \rangle$. By assumption, $V^g \cap H \leq K$ for all $g \in G$, and as S is a splitting field of H, Theorem 3.3 therefore allows us to assume that $H = G$. Let L be an indecomposable $R[V \times K]$-module such that $M | L^{\uparrow H}$. The statement certainly follows if $L^{\uparrow H}$ is indecomposable. This in turn follows from the proof of the following

Lemma 3.7. Let $H = \langle x \rangle$ with Sylow p-subgroup Q and complement T. Let (F, R, S) be a p-modular system and assume S is a splitting field of $S[H]$. Then

i) Any indecomposable $F[H]$-module is uniserial.

ii) An $R[H]$-module M is indecomposable if and only if $M/M(\pi)$ is indecomposable.

Proof: Let $1 = \sum_{i=1}^{|T|} e_i$ be the primitive idempotent decomposition of 1 in $S[T]$. Then $e_i \in R[T]$ for all i, as $(p, |T|) = 1$. Denote the corresponding idempotent in $F[T]$ by $\bar{e}_i$. Let $\bar{N}$ be an indecomposable $F[Q]$-module. Then $\bar{N}$ is uniserial as we saw in the example of Section I, 8. Moreover $\bar{N}^{\uparrow H}$ decomposes into

$$\bar{N}^{\uparrow H} = \bigoplus_{i=1}^{|T|} (\bar{N}^{\uparrow H})\bar{e}_i$$

as an $F[H]$-module, and $(\bar{N}^{\uparrow H}\bar{e}_i)_{\downarrow Q} \simeq \bar{N}$ as H is cyclic. Thus $(\bar{N}^{\uparrow H})\bar{e}_i$ is uniserial by Corollary I.9.6. As any $F[H]$-module is $F[Q]$-projective, i) follows.

To see ii), assume first that $T = 1$. Then the statement immediately follows from the Jordan Normal Form. But then the general statement follows by the argument above, using $1 = \sum e_i$.

To finish the proof above, it suffices to prove that $(L^{\uparrow H})_{\downarrow Q}$

is indecomposable. Let N be an R[V]-module such that $L | N^{\uparrow V\times T}$. Then $(L^{\uparrow H})_{\downarrow Q} | ((N^{\uparrow Q})^{\uparrow H})_{\downarrow Q}$. But $N^{\uparrow Q}$ is indecomposable by Example 1 of Section 1. Thus $L^{\uparrow H} \simeq (N^{\uparrow H})e_i$ by the argument above and $(L^{\uparrow H})_{\downarrow Q} \simeq N^{\uparrow Q}$.

Remark. What we have just proved is very important for the proof of Brauer's Second Main Theorem. It is actually a special case of a more general result due to Green (see Theorem 11.10).

It is possible to prove the results of Lemmas 3.1 through 3.5 in a different way, namely via Corollary 1.9, in view of Theorem 2.3. Let us explain this in more detail:

Lemma 3.8. Let M be an $\mathcal{O}[G]$-module, and let X be an indecomposable direct summand. Then X is H-projective for some $H \leq G$ if and only if the corresponding idempotent of $(M, M)^G$ is H-projective.

Proof: Clear by Theorem 2.3, as any isomorphism $X \longrightarrow X$ will factor through X, only.

But then the next question is: If $e \in (M, M)^G$ is a primitive idempotent, how do we prove that $e \in (M, M)^G_U$ for some element $U \in \mathcal{H}$? This is precisely the situation we deal with in the following useful

Lemma 3.9 (Rosenberg's lemma) (see Rosenberg (1961)). Let A be a ring and $e \in A$ an idempotent such that eAe is a local ring. Assume furthermore that $e \in \Sigma \mathfrak{a}_i$, where $\mathfrak{a}_i$ is an ideal in A. Then there exists an i_0 such that $e \in \mathfrak{a}_{i_0}$.

Proof: Multiplying from left and right by e, we may as well assume that $A = eAe$. Thus A is local, and not all $\mathfrak{a}_i$'s are contained in the maximal ideal of A. Hence $\mathfrak{a}_{i_0} = A$ for some i_0.

Remark: We have chosen the basic assumption of the lemma so that the proof is as obvious as possible. Observe that if (F, R, S) is a p-modular system and A is a finite dimensional F-algebra or R-order, the assumption above holds for A, as we saw in Ch. I, Sec. 5 & 11.

Let us demonstrate the effectiveness of the strategy just suggested by proving the following remarkable result on vertices. It was obtained recently by Burry & Carlson (1982) and independently by Puig (1981).

Theorem 3.10. Let (F, R, S) be a p-modular system, and choose Θ equal F or R. Let $V \leq G$ and $N_G(V) \leq H \leq G$. Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable and assume $M_{\downarrow H}$ has an indecomposable direct summand with V as vertex. Then M has V as vertex.

Proof: Let $N | M_{\downarrow H}$ be indecomposable with vertex V and $e \in (M, M)^H_V$ be the corresponding idempotent. Let $\alpha \in (M, M)^V$ such that $e = \mathrm{Tr}_V^H(\alpha)$ and set $\varepsilon = \mathrm{Tr}_V^G(\alpha)$. Then $\varepsilon = e \bmod (M, M)^H_{\mathfrak{Y}}$ by Corollary 1.9 i), where $\mathfrak{Y} = \{V^\gamma \cap H | \gamma \notin H\}$. Moreover, $e \not\equiv 0 \bmod (M, M)^H_{\mathfrak{Y}}$ by Lemma 3.4 and Rosenberg's Lemma, as by assumption V is a vertex of N and $V \notin \mathfrak{Y}$. But $(M, M)^G$, and hence

$$(5) \qquad (\overline{M, M})^G := (M, M)^G/(M, M)^H_{\mathfrak{Y}} \cap (M, M)^G$$

is a local ring. As $\bar{\varepsilon} \in (\overline{M, M})^G$ is an idempotent, ε is invertible, and thus

(6) $$(M, M)^G = (M, M)^G\varepsilon \subseteq (M, M)^G_V$$

as $(M, M)^G_V$ is an ideal in $(M, M)^G$. Hence M is V-projective by Lemma 3.8, and the theorem follows from Theorem 3.3 i).

We end this section with

Definition 3.11. Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable with V as vertex. Then an indecomposable module $U \in \mathbf{M}_\Theta(V)$ with $M | U^{\uparrow G}$ is called a source of M.

Remark: If U is a source of M, then so is $U \otimes g$ for any $g \in N_G(V)$, obviously, and with V as vertex. In general $U \otimes g$ is a source of M and V^g as vertex, for any $g \in G$. We shall prove the converse:

Theorem 3.12 (Green). Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable with V as vertex, and let M be H-projective for some $H \leq G$. Then

i) If $U_1, U_2 \in \mathbf{M}_\Theta(V)$ are both sources of M, then $U_1 \sim U_2 \otimes g$ for some $g \in N_G(V)$.

ii) Let N be any indecomposable component of $M_{\downarrow H}$ with vertex $W \underset{G}{=} V$. Then there exists an indecomposable module in $\mathbf{M}_\Theta(W)$ which is a common source for M and N.

Proof: Let $U \in \mathbf{M}_\Theta(V)$ be indecomposable such that $U | M_{\downarrow V}$ and $M | U^{\uparrow G}$. Then Mackey decomposition yields a $\gamma_i \in G$ such that

(7) $$U \,|\, ((U_i \otimes \gamma_i)_{\downarrow \gamma_i^{-1} V \gamma_i \cap V})^{\uparrow V}$$

for $i = 1, 2$. As the vertex of U is V, this forces $V^{\gamma_i} = V$.

Consequently, $U \approx U_i \otimes \gamma_i$ as the latter is indecomposable for $i = 1, 2$ and i) follows.

Finally, Mackey decomposition proves that $N \mid ((U_1 \otimes \gamma)_{\downarrow \gamma^{-1}V\gamma \cap H})^{\uparrow H}$ for some γ. Hence $\gamma^{-1}V\gamma \subset H$, and $U_1 \otimes \gamma$ is a common source.

Example 3. With reference to Example 1, M^*, ΩM and $\mho M$ has U^*, ΩU and $\mho U$ resp. as sources, with the notation of Definition 3.11.

It is natural at this stage to ask about connections between relative projectivity of an indecomposable $R[G]$-module M and the corresponding $F[G]$-module $\overline{M} = M/M(\pi)$, where (F, R, S) is a p-modular system and (π) is the maximal ideal of R. The answer is nothing much, except the obvious.

Lemma 3.13. Same notation as above. Let $H \leq G$. Then

i) Let $N \in \mathbf{M}_\Theta(H)$. Then $\overline{N^{\uparrow G}} = \overline{N}^{\uparrow G}$.

ii) Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable. Assume M is H-projective. Then so is $\overline{M}$.

Proof: i) is by definition, and ii) follows from i).

Remark: The coverse of ii) above is not always true. For an example, see Feit (1982), p. 111.

This has the following straightforward but useful application to relative projective homomorphisms.

Lemma 3.14. Let $A, B \in \mathbf{M}_R(G)$, and set $\overline{A} = A/A\pi$, $\overline{B} = B/B\pi$. For $\phi \in (A, B)^G$, the induced map in $(\overline{A}, \overline{B})^G$ is denoted by $\overline{\Phi}$. Let $\mathcal{W}$ be any family of subgroups of G. Then

(8) $$((A+B)^G_{\mathcal{W}} + (A, B)^G)/(A, B)^{G}\pi \subseteq (\bar{A}, \bar{B})^G_{\mathcal{W}} .$$

Proof: By Theorem 2.3 and Lemma 3.14.

Remark: As we saw in Corollary 2.9, equality actually holds in (8) if $\mathcal{W} = \{1\}$.

4. Green Correspondence.

We are now able to present another major result on restriction and induction of modules, namely Green Correspondence.

We continue to let Θ be a principal ideal domain such that Krull-Schmidt holds for modules in $\mathbf{M}_\Theta(H)$ for all $H \leq G$. Moreover, we let V be a fixed subgroup of G. We shall be concerned with indecomposable $\Theta[G]$-modules with V as vertex. We also consider some arbitrary, but fixed $H \geq N_G(V)$, and introduce the following standard notation

(1) $$\begin{aligned} \mathfrak{X} &= \{W \leq G \mid W \leq V^g \cap V,\ g \in G \setminus H\} \\ \mathfrak{Y} &= \{W \leq G \mid W \leq V^g \cap H,\ g \in G \setminus H\} \end{aligned}$$

Theorem 4.1 (Green Correspondence) (Green (1964)). There is a one-to-one correspondence between indecomposable modules in $\mathbf{M}_\Theta(G)$ with V as vertex and indecomposable modules in $\mathbf{M}_\Theta(H)$ with V as vertex, which is characterized as follows:

i) Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable with V as vertex. Then $M_{\downarrow H}$ has a unique indecomposable direct summand $f(M)$ with V as vertex. Furthermore,

(2) $$M_{\downarrow H} \simeq f(M) \oplus (\bigoplus_i N_i)$$

where the vertices of N_i all lie in $\mathfrak{Y}$, for all i.

ii) Let $N \in \mathbf{M}_\Theta(H)$ be indecomposable with V as vertex. Then $N^{\uparrow G}$ has a unique indecomposable direct summand $g(N)$ with V as vertex. Furthermore

(3) $$N^{\uparrow G} \simeq g(N) \oplus (\bigoplus_i M_i)$$

where M_i has a vertex in $\mathfrak{X}$, for all i.

iii) In particular, $g(f(M)) = M$ and $f(g(N)) = N$.

Proof: i) Let M be given as in i). We first claim there exists $N | M_{\downarrow H}$ indecomposable with vertex V such that $M | N^{\uparrow G}$. Indeed, if $U \in \mathbf{M}_\Theta(V)$ is a source of M, there exists an indecomposable direct summand N of $U^{\uparrow H}$ such that $M | N^{\uparrow H}$. It immediately follows that N has V as vertex. Moreover $M_{\downarrow H} | (N^{\uparrow G})_{\downarrow H}$ has an indecomposable direct summand with V as vertex, by Theorem 3.3 iii). But only N will do by Lemma 3.1. Now all of i) follows from Lemma 3.1.

ii) Given N as in ii), choose $M | N^{\uparrow G}$ indecomposable such that $N | M_{\downarrow H}$. Let $N^{\uparrow G} = M \oplus (\oplus M_i)$, where M_i is indecomposable for all i. As M is V-projective, then, Lemma 3.1 and Theorem 3.3 i) yield that in fact V is a vertex of M. Now all of ii) follows from Lemma 3.4.

Remark: Notice that elements in $\mathfrak{X}$ are all properly contained in V. In particular, they have order less than the order V. This is not true for elements of $\mathfrak{Y}$, although of course $V \notin \mathfrak{Y}$. For this reason, we point out that

Lemma 4.2. Same notation as in Theorem 4.1. Consider $\{V^g | V^g \leq H\}$ and let $\{V^{g_j} | j \in J\}$ be a set of representatives of the H-orbits of this set. Then $M_{\downarrow H}$ has an indecomposable direct summand N_j with V^{g_j} as vertex. In particular, $M_{\downarrow H}$ has at least $|J|$ non-isomorphic direct summands.

Proof: As M has V^g as vertex for all $V^g \leq H$, $M_{\downarrow H}$ has an indecomposable direct summand with V^g as vertex by Theorem 3.3 iii). Let N_j be a direct summand of $M_{\downarrow H}$ with vertex V^{g_j}. As their vertices are not H-conjugate, they cannot be isomorphic by Theorem 3.3 i).

In Ch. III, Sec. 8, we shall need the following consequence of Green Correspondence (see Alperin (1981)). By a p-local $\Theta[G]$-module we mean a direct sum of modules induced from subgroups of the form $N_G(V)$, where $V \neq 1$ is a p-group.

Lemma 4.3. Let $M \in \mathbf{M}_\Theta(G)$. Then there exists p-local $\Theta[G]$-modules L_1 and L_2, and projective $\Theta[G]$-modules P_1 and P_2 such that $M \oplus L_1 \oplus P_1 \simeq L_2 \oplus P_2$.

Proof: We may assume that M is indecomposable and non-projective. Let V be a vertex of M and $f(M)$ the Green Correspondent of M in $N_G(V)$. Then $f(M)^{\uparrow G} \simeq M \oplus M'$, where M' is a sum of modules with smaller vertices. So by induction, the result holds for M'. As $f(M)^{\uparrow G}$ is p-local, we are done.

For completeness, we also notice:

Lemma 4.4. Assume we are in the situation of Theorem 4.1.

Then

i) Green Correspondence commutes with taking dual modules.

ii) Green Correspondence commutes with the Heller operators.

Proof: Easy exercise. Use Theorem I.6.4 for i).

Remark: Notice that our proof of Green Correspondence is based on Theorem 3.3 and Lemmas 3.1 & 4, and one more observation, which is easy to prove if $N_G(V) \leq H$, namely the fact mentioned in the remark following the proof of Theorem 3.3. As promised there, we now prove that this holds without the restriction that $N_G(V)$ be contained in H.

Theorem 4.5 (Burry (1979)). Let $M \in \mathbf{M}_\Theta(G)$ be indecomposable with V as a vertex and let $V \leq H$. Then there exists $N \in \mathbf{M}_\Theta(H)$ indecomposable with V as vertex as well such that

i) $N | M_{\downarrow H}$.

ii) $M | N^{\uparrow G}$.

Proof: Let $K = N_G(V)$. Let $f_1(M)$ be the Green Correspondent of M in $M_{\downarrow K}$. By Corollary 2.4 ii) let $L | f_1(M)_{\downarrow K \cap H}$ be indecomposable with $f_1(M) | L^{\uparrow K}$. Then L has V as a vertex by Theorem 3.3 i), as $V \trianglelefteq K$. Let $g_2(L)$ be the Green Correspondent of L in $L^{\uparrow H}$. We now claim that $N = g_2(L)$ will do in the theorem. Indeed, N has V as vertex, by choice. Also, $M | L^{\uparrow G}$ by Green Correspondence. Moreover, $L^{\uparrow H} = N \oplus (\bigoplus_i X_i)$ where each X_i has vertex properly contained in V, again by Green Correspondence. Thus ii) holds.

Moreover, $L | M_{\downarrow K \cap H}$, again by choice. Hence there exists an indecomposable direct summand N_1 of $M_{\downarrow H}$ such that $L | N_{1 \downarrow H \cap K}$.

As M and L both have V as vertex, Theorem 3.3 i) asserts that N_1 has V as vertex, too. Hence $N_1 = N$ by Green Correspondence, and i) holds as well.

Finally, Theorem 3.10 gives the following improvement of Green Correspondence:

Theorem 4.6 (Burry & Carlson (1982)). Let $A \in \mathbf{M}_{\Theta}(G)$ be indecomposable with V as vertex. Let $N_G(V) \leq H$, and let $f(A)$ be the Green Correspondent of A in $A_{\downarrow H}$.

Let $B \in \mathbf{M}_{\Theta}(G)$ be arbitrary. Then the following are equivalent:

i) $A|B$

ii) $f(A)|B_{\downarrow H}$.

Proof: Obvious.

Remark: Needless to say, Green Correspondence has a number of applications, as we will see when we proceed. However, it should also be pointed out what Green Correspondence does not do. For instance, it does not tell us if the restriction or induction of a module decomposes at all. Naturally we would like to say something about the Green Correspondents of the simple $F[G]$-modules. But if G is a group with a (B, N)-pair and $B = U.H$ where $U \in \mathrm{Syl}_p(G)$ and char $F = p$, a class of groups which has been examined in great detail (and include all groups of Lie type) starting with Curtis (1965) & (1970) and Richen (1969), it turns out that the restriction of any simple $F[G]$-module to B is indecomposable. Nevertheless, the restrictions have a remarkable property. Namely, the socle and the head are both 1-dimensional. However, if we then turn to other simple groups, or other characteristics for groups with a (B, N)-pair, the Green Correspondents of the simple modules may be rather small, but they do not seem to have particularly nice or remarkable properties. For some non-trivial examples, see Erdmann (1977a) and (1979), Landrock & Michler (1978) and (1980) and Schneider (1983a).

5. Relative projective homomorphisms.

Inspired by the success of the previous section we now concentrate on relatively projective homomorphisms. However, as we discovered in Section 1, it is easier first to prove the results for the trace map between modules.

In the following we only require Θ to be a principal ideal domain. For $X \subseteq G$, we set $[X] = \sum_{x \in X} x$.

Lemma 5.1. Let $V \leq H \leq G$, and let $N \in \mathbf{M}_{\Theta}(H)$. Then

(1) $$Tr_H^G(N_V^H) \subseteq (N\uparrow_H^G)_V^G \ .$$

In particular, $(N\uparrow_H^G)^G = (N\uparrow_H^G)_H^G$.

Proof: Let $a \in N_V^H$. Then $a = \alpha[V\backslash H]$ for some $\alpha \in N^V$, where as usual $V\backslash H$ denotes an arbitrary right transversal of V in H. Thus

(2) $$\begin{aligned} Tr_H^G(a) &= \sum_{g_i \in H\backslash G} \alpha[V\backslash H] \otimes g_i \\ &= \alpha[V\backslash H] \otimes [H\backslash G] \\ &= \alpha \otimes [V\backslash G] \end{aligned}$$

which belongs to $(N\uparrow_H^G)_V^G$.

This has the following important consequence. For $H \leq G$ and $\mathcal{H}$ a family of subgroups of G, set

(3) $$\mathcal{H}_G \cap H = \{U^g \cap H \mid U \in \mathcal{H}\} \quad .$$

Lemma 5.2. With the notation above,

(4) $$N^H_{\mathcal{H} \cap_G H} \simeq (N\uparrow^G_H)^G_{\mathcal{H}} = (N\uparrow^G_H)^G_{\mathcal{H} \cap_G H} \quad ,$$

where Tr^G_H provides the isomorphism.

Proof: Let $U \in \mathcal{H}$, $g \in G$. Set $V = U^g \cap H$. Then (1) holds for this choice of V, and thus

(5) $$Tr(N^H_{\mathcal{H} \cap_G H}) \subseteq (N\uparrow^G_H)^G_{\mathcal{H} \cap_G H} \quad .$$

Conversely, let $x = \sum a_i \otimes g_i \in (N\uparrow^G_H)^G_U$. Then, by Mackey decomposition, as $x \in (N\uparrow^G_H)^U$,

(6) $$x = (\sum_{g_j \in H \backslash G/U} a_j \otimes g_j [H^{g_j} \cap U \backslash U])[U \backslash G]$$

$$= (\sum_j a_j \otimes [H \cap U^{g_j^{-1}} \backslash U^{g_j^{-1}}] g_j)[U \backslash G]$$

$$= (\sum_j a_j \otimes [H \cap U^{g_j^{-1}} \backslash U^{g_j^{-1}}])[U^{g_j^{-1}} \backslash G]$$

as $xg_j^{-1} = x$,

$$= \sum_j a_j \otimes [H \cap H^{g_j^{-1}} \backslash G]$$

$$= (\sum_j a_j \otimes [H \cap U^{g_j^{-1}} \backslash H])[H \backslash G]$$

which belongs to $Tr^G_H(N^H_{\mathcal{H} \cap_G H})$, and (4) follows.

Corollary 5.3. Let $M \in \mathbf{M}_{\Theta}(G)$ and assume M is H-projective. Then

$$M^G_{\mathcal{H}} = M^G_{\mathcal{H} \cap_G H} \quad . \tag{7}$$

Proof: This follows from (4), as $(M_{\downarrow H})^{\uparrow G} \approx M \oplus W$ for some $W \in \mathbf{M}_{\Theta}(G)$ by assumption.

For completeness, and in view of the heading of this section, we state Lemma 5.2 for homomorphisms.

Corollary 5.4 (Knörr (1979)). Let $H \leq G$, and let $M \in \mathbf{M}_{\Theta}(G)$, $N \in \mathbf{M}_{\Theta}(H)$. Let $\mathcal{H}$ be any family of subgroups of G. Then

i) Let $\phi \in (M, N)^H$. Then $Tr^G_H(\phi) \in (M, N\uparrow^G_H)^H$ is $\mathcal{H}$-projective if and only if ϕ is $\mathcal{H} \cap_G H$-projective.

ii) Let $\phi \in (N, M)^H$. Then $Fr^G_H(\phi) \in (N\uparrow^G_H, M)^G$ is $\mathcal{H}$-projective if and only if ϕ is $\mathcal{H} \cap_G H$-projective.

Furthermore we may now prove the following refinement of Corollary 1.9.

Theorem 5.5. Assume Krull-Schmidt holds for elements in $\mathbf{M}_{\Theta}(H)$ for all $H \leq G$. Let $M \in \mathbf{M}_{\Theta}(G)$, and let $V \leq G$, $N_G(V) \leq H \leq G$. Define

$$\mathfrak{X} = \{V^{\gamma} \cap V \mid \gamma \notin H\}, \quad \mathfrak{Y} = \{V^{\gamma} \cap H \mid \gamma \notin H\} \quad . \tag{8}$$

Assume moreover that M is V-projective. Then

i) $M^H = M^G + M^H_{\mathfrak{Y}}$.

ii) The Trace Map induces an Θ-isomorphism

(9) $$M^{\mathfrak{Y},H} \simeq M^{\mathfrak{X},G} .$$

iii) $M^G_{\mathfrak{X}} = M^H_{\mathfrak{Y}} \cap M^G$.

Proof: i) One inclusion is obvious. Moreover, $M^H = M^H_V$ and $M^G = M^G_V$ by Corollary 5.3. Hence the other inclusion follows from Corollary 1.9 ii).

iii) is an immediate consequence of i) and ii).

Thus it remains to prove ii). This will follow from

Lemma 5.6. Same notation as above. Let $L \in \mathbf{M}_\Theta(H)$ be indecomposable and V-projective such that $M | L{\uparrow}^G_H$. Then

i) $L^H_{\mathfrak{X}} = L^H_{\mathfrak{Y}} = L^H_{\mathfrak{X} \cap {}_G H}$.

ii) Either M is $\mathfrak{X}$-projective, or

(10) $$L^{\mathfrak{X},H} \simeq M^{\mathfrak{X},G} .$$

Proof: We have

(11) $$L^H_{\mathfrak{X}} \subseteq L^H_{\mathfrak{X} \cap_G H} \subseteq L^H_{\mathfrak{Y}} = L^H_{\mathfrak{X} \cap_H V} = L^H_{\mathfrak{X}}$$

as $\mathfrak{X} \cap {}_G H \subseteq \mathfrak{Y} = \mathfrak{X} \cap {}_H V$, while the last equality follows from Corollary 5.3.

ii) (Note that we do not necessarily know that $L | M_{\downarrow H}$). By assumption, $L^{\uparrow G} = M \oplus M'$ for some $M' \in \mathbf{M}_\Theta(G)$. Moreover, either M or M' is $\mathfrak{X}$-projective by Lemma 3.4. Also, $(L^{\uparrow G})_{\downarrow H} \simeq L \oplus L'$, where $L' \in \mathbf{M}_\Theta(H)$ is $\mathfrak{Y}$-projective by Lemma 3.1. Assume now that M is not $\mathfrak{X}$-projective. Then Corollary 5.3 and Lemma 5.2 imply that

(12) $$M^G/M^G_{\mathfrak{X}} \simeq (L^{\uparrow G})^G/(L^{\uparrow G})^G_{\mathfrak{X}} \simeq L^H/L^H_{\mathfrak{X} \cap_G H}$$

and we are done by i).

Thus we moreover have that

(13) $$M^G/M^G_{\mathfrak{X}} \simeq L^H/L^H_{\mathfrak{Y}} \simeq (L^{\uparrow G})^H/(L^{\uparrow G})^H_{\mathfrak{Y}}$$

by the remark above on $(L^{\uparrow G})_{\downarrow H}$. Thus

(14) $$M^G/M^G_{\mathfrak{X}} \simeq M^H/M^H_{\mathfrak{Y}}$$

all under the assumption that M is not $\mathfrak{X}$-projective. However if in fact M is $\mathfrak{X}$-projective then ii) of Theorem 5.5 is in fact vacuous by Mackey decomposition. Thus Theorem 5.5 ii) always holds.

Again we state these results for homomorphisms.

Corollary 5.7. Same notation as in Theorem 5.5. Let $A, B \in \mathbf{M}_{\Theta}(G)$ and assume furthermore that A or B is V-projective. Then

i) $(A, B)^H = (A, B)^G + (A, B)^{\mathfrak{Y}, H}$.

ii) The trace map induces an Θ-isomorphism

(15) $$(A, B)^{\mathfrak{Y}, H} \simeq (A, B)^{\mathfrak{X}, G}$$

and if $A = B$, this in fact is a ring isomorphism.

iii) $(A, B)^G_{\mathfrak{X}} = (A, B)^H_{\mathfrak{Y}} \cap (A, B)^G$.

Proof: By Lemma 1.2, (A, B) is V-projective if A or B is. Furthermore Corollary 1.10 asserts that the induced map of (15) in fact is a ring isomorphism if $A = B$.

Using Green Correspondence, we now get the following result by Feit (1969). See also Green (1972).

Corollary 5.8. Same notation as above. Assume that, say B is indecomposable with V as vertex, and let f denote Green Correspondence from G to H w.r.t. V. Then the trace map induces an Θ-isomorphism

(16) $$(A, B)^{\mathfrak{X},G} \simeq (A, f(B))^{\mathfrak{X},H} .$$

(Of course, we get a similar result if instead A is assumed to have V as vertex). If both A and B are indecomposable with V as vertex,

(17) $$(A, B)^{\mathfrak{X},G} \simeq (f(A), f(B))^{\mathfrak{X},H}$$

and (17) is a ring isomorphism if $A = B$.

Proof: By Lemma 5.6.

Corollary 5.9. Same notation and assumptions as above. Assume furthermore that $\mathfrak{X} = \{1\}$. Then

(18) $$\mathrm{Ext}^1_{F[G]}(A, B) \simeq \mathrm{Ext}^1_{F[H]}(f(A), f(B)) .$$

Proof: Clear.

6. Tensor products.

Before we start elaborating on the results we developed so far, we remind the reader of the following fundamental results on tensor product of $\Theta[G]$, where Θ is a principal ideal domain. If $A, B \in \mathbf{M}_\Theta(G)$, the tensor product over Θ will simply be denoted by $A \otimes B$. Now $\Theta[G]$ acts on $A \otimes B$ by diagonal action, i.e.,

$(a \otimes b)g = ag \otimes bg$ for $a \in A$, $b \in B$ and $g \in G$. Thus $A \otimes B \in \mathbf{M}_\Theta(G)$.

Lemma 6.1. Let $A, B \in \mathbf{M}_\Theta(G)$. Then

i) $B \otimes A^* \simeq (A, B)$

ii) $A^* \otimes B^* \simeq (A \otimes B)^*$

iii) $A \otimes B \simeq B \otimes A$

as $\Theta[G]$-modules.

Proof: i) Recall that the map $\Delta : B \otimes A^* \longrightarrow (A, B)$ given by

(1) $$b \otimes a^* \longrightarrow \phi_{b,a^*} : x \longrightarrow a^*(x)b$$

and extended by linearity is an Θ-isomorphism. Now let $g \in G$ be arbitrary. Then

(2) $$\begin{aligned} \phi_{bg,a^*g}(x) &= [a^*g](x)bg \\ &= a^*(xg^{-1})bg \\ &= \phi_{b,a^*}(xg^{-1})g \\ &= [\phi_{b,a^*}g](x) \quad . \end{aligned}$$

ii) Again there is a natural Θ-isomorphism of $A^* \otimes B^*$ onto $(A \otimes B)^*$, namely, if $\alpha \in A^*$ and $\beta^* \in B^*$, the corresponding element $\phi_{\alpha \otimes \beta} \in (A \otimes B)^*$ is given by

(3) $$\phi_{\alpha \otimes \beta}(a \otimes b) = \alpha(a)\beta(b) \quad .$$

Again we must check G-linearity. Let $g \in G$. Then

$$(4) \qquad \phi_{(\alpha \otimes \beta)g}(a \otimes b) = \alpha(ag^{-1})\beta(bg^{-1}) = \phi_{\alpha \otimes \beta}(ag^{-1} \otimes bg^{-1}) = \phi_{\alpha \otimes \beta}((a \otimes b)g^{-1}) = [\phi_{\alpha \otimes \beta}g](a \otimes b) \quad .$$

iii) is trivial.

Corollary 6.2. With the same notation, we furthermore have that

i) $(A, B) \simeq (B^*, A^*)$

ii) $(A, B)^* \simeq (B, A) \simeq (A^*, B^*)$

as $\Theta[G]$-modules.

Proof: Clear.

Corollary 6.3. Let $H \leq G$, and let $M \in \mathbf{M}_\Theta(G)$, $N \in \mathbf{M}_\Theta(H)$. Then

$$(5) \qquad M \otimes (N\uparrow_H^G) \simeq (M_{\downarrow H} \otimes N)\uparrow_H^G$$

as $\Theta[G]$-modules.

Proof: By Lemmas 1.2 and 6.1.

Corollary 6.4 (Mackey Tensor Product Decomposition). Let $K, H \leq G$ and let $A \in \mathbf{M}_\Theta(K)$, $B \in \mathbf{M}_\Theta(H)$. Then

$$(6) \qquad A\uparrow_K^G \otimes B\uparrow_H^G \simeq \bigoplus_{g_i \in K \backslash G/H} ((A \otimes g_i)_{\downarrow K^{g_i} \cap H} \otimes B_{\downarrow K^{g_i} \cap H})\uparrow^G \quad .$$

Proof: Apply Corollary 6.3 twice.

In the following we let (F, R, S) be a p-modular system and assume for convenience that Θ is always F or R in order to assure that Krull-Schmidt holds.

Corollary 6.5. Let $A, B \in \mathbf{M}_\Theta(G)$. Assume A or B is H-projective, where $H \leq G$. Then $A \otimes B$ is H-projective as well.

In particular, $A \otimes B$ is projective whenever A or B is.

Proof: Let by assumption $A | L^{\uparrow G}$, where $L \in \mathbf{M}_\Theta(H)$. Then

$$(7) \qquad A \otimes B | L^{\uparrow G} \otimes B \simeq (L \otimes B_{\downarrow H})^{\uparrow G}$$

by Corollary 6.3, which proves the statement.

As an important application of this, we state

Corollary 6.6. Let $A \in \mathbf{M}_\Theta(G)$ be H-projective and let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of elements in $\mathbf{M}_\Theta(G)$ (in particular, Z is assumed to be Θ-free), whose restriction to $\Theta[H]$ splits. Then

$$(8) \qquad 0 \to X \otimes A \to Y \otimes A \to Z \otimes A \to 0$$

splits.

Proof: By Corollary 6.5 and 2.4 v).

Lemma 6.7. Let $Q \in Syl_p(G)$ be normal in G. Let E be any simple $F[G]$-module. Let I denote the trivial $F[G]$-module. Then

$$(9) \qquad P_E = P_I \otimes E$$

where as usual the projective cover P_M of $M \in \mathbf{M}_\Theta(G)$ denotes the projective cover of M.

Proof: By Corollary I.9.6, $\mathrm{Soc}(P_E) = \mathrm{Soc}(P_{E\downarrow Q})$. Hence $\dim_F(P_E) = |Q|\dim_F(E)$ by Proposition I.9.3. By Corollary 6.5, $P_I \otimes E$ is projective, and $I \otimes E = E$ is a submodule. Hence in fact $P_E | P_I \otimes E$ and (9) must hold, as the two modules have the same dimension.

Example 1. See Example 3 of Section I.18.

Example 2. Let $G \simeq Q \times K$, where $Q \in \mathrm{Syl}_p(G)$. For any simple $F[K]$-module E, let E denote the inflation to G of E, as well. This obviously describes all the simple $F[G]$-modules, as $Q \triangleleft G$ (cf. Proposition I.9.2). The projective cover of any such simple $F[G]$-module E is just $(E_{\downarrow K})^{\uparrow G}$.

Indeed in view of Lemma 6.7, it suffices to prove this for the trivial $F[G]$-module I. But as $(I_{\downarrow K})^{\uparrow G}$ has dimension $|Q|$, $(I_{\downarrow K})^{\uparrow G}$ is not only projective, but indecomposable as well. Finally, the Nakayama relations assert that

$$(10) \qquad ((I_{\downarrow K})^{\uparrow G}, I)^G \simeq (I_{\downarrow K}, I_{\downarrow K})^K \simeq F \quad .$$

Thus $(I_{\downarrow K})^{\uparrow G} \simeq P_I$.

We now elaborate a little further on Lemma 6.1. Although the following result is trivial to prove, it is so important that we call it a theorem. Also Lemma 6.1 may be thought of as a special case of this. Again we just need $\mathcal{O}$ to be a principal ideal domain.

Theorem 6.8. Let $A, B, C \in \mathbf{M}_{\mathcal{O}}(G)$. Then

$$(11) \qquad (A \otimes B, C) \simeq (A, B^* \otimes C)$$

as $\Theta[G]$-modules.

Proof: This follows immediately from Lemma 6.1 i) and ii).

Corollary 6.9. Same notation as above. Then

(12) $$(A \otimes B,\ C)^G \simeq (A,\ B^* \otimes C)^G\ .$$

Proof: Clear.

These results open a new way to discuss relative projectivity. First we have

Theorem 6.10 (Landrock and Michler ( )). Same notation as above. Let $\mathcal{H}$ be any family of subgroups of G. Then

(13) $$(A \otimes B,\ C)^G_{\mathcal{H}} \simeq (A,\ B^* \otimes C)^G_{\mathcal{H}}$$

as $\Theta[G]$-modules.

Proof: Let Ψ denote the isomorphism in (11). In particular, Ψ induces an isomorphism

(14) $$(A \otimes B,\ C)^H \simeq (A,\ B^* \otimes C)^H$$

for all $H \leq G$. Now, let $U \in \mathcal{H}$ be arbitrary, and let $\phi \in (A \otimes B,\ C)^G_U$, say $\phi = Tr^G_U(\lambda)$, where $\lambda \in (A \otimes B,\ C)^U$. Then

(15) $$\Psi(\phi) = \Psi(Tr^G_H(\lambda)) = \Psi(\sum_{g \in H \backslash G} \lambda g) = \sum_{g \in H \backslash G} \Psi(\lambda)g = Tr^G_H(\Psi(\lambda))$$

simply because Ψ is an $\Theta[G]$-homomorphism. Hence Ψ maps $(A \otimes B,\ C)^G_U$ into $(A,\ B^* \otimes C)^G_U$, and as Ψ is an isomorphism, (14) follows with $\mathcal{W} = \{U\}$ by duality. Adding over all $U \in \mathcal{W}$, we get (14) in general, then.

To consider applications of this, let (F, R, S) be a p-modular system.

Corollary 6.11. Let $A, B \in \mathbf{M}_F(G)$, and let P_I denote the projective cover of the trivial F[G]-module I. Then

$$(15) \qquad \dim_F((A, B)_1^G) = \dim_F((A^* \otimes B)(\sum_{g \in G} g))$$

which also equals the multiplicity of P_I as a direct summand of $A^* \otimes B$.

In particular,

$$(17) \qquad \dim_F(A)\dim_F(B) \geq \dim_F((A, B)_1^G)\dim_F(P_I) \quad .$$

Proof: By Theorem 6.8,

$$(18) \qquad (A, B)_1^G \simeq (I, A^* \otimes B)_1^G \quad .$$

Let $A^* \otimes B = \bigoplus_i L_i$, where L_i is indecomposable for all i. Then

$$(19) \qquad (A, B)_1^G \simeq \bigoplus_i (I, L_i)_1^G \quad .$$

But now Lemma 2.7 yields that $(1, L_i)_1^G \neq 0$ if and only if $L_i \simeq P_I$. Or equivalently, if and only if $L_i(\sum_{g \in G}) \simeq F$, as P_I occurs with multiplicity one in $F[G]_{F[G]}$ and the socle of P_I is spanned by the norm element $\sum_{g \in G} g$.

Corollary 6.12 (Feit (1969). Let S be a splitting field of S[G], and let χ be a character of S[G]. Let $Q \in Syl_p(G)$, and assume $Q \cap Q^g = 1$ for all $g \notin N_G(Q)$. Let $N_G(Q) \leq H$ and assume

$\chi(1)^2 \leq |Q|$. The

(20) $$(\chi, \chi)_G = (\chi, \chi)_H \quad .$$

Proof: Let M be an R-form of χ. Then

(21) $$(\chi, \chi)_G = \text{rank}_R((M, M)^G)$$
$$= \text{rank}_R((I, M^* \otimes M)^G)$$

where I denotes the trivial R[G]-module. Now comes the magic step. We claim that

(22) $$\text{rank}_R((I, M^* \otimes M)^G) = \text{rank}_R((I, M^* \otimes M)^{1,G}) \quad .$$

Indeed what we claim is that if $\phi \in (I, M^* \otimes M)^{1,G}$ then $\phi(I) \subseteq (M^* \otimes M)\pi$. If not, then $\phi(I)$ is an R-pure submodule of $M^* \otimes M$, and reducing modulo π induces a non-trivial homomorphism $\overline{\phi} \in (\overline{I}, \overline{M}^* \otimes \overline{M})^G$, where $\overline{X} := X/X\pi$ for $X \in \mathbf{M}_\Theta(G)$. By Corollary 2.9, $\overline{\phi}$ is projective as well, a contradiction by Corollary 6.11 as $(M, M)_1^G \subsetneq (M, M)^G$. But now, Theorem 5.5 yields that

(23) $$(I, M^* \otimes M)^{1,G} \simeq (I, M^* \otimes M)^{1,H}$$

and going through the same step of arguments as above, only this time in reverse order, we get (20).

Remark: To prove this result, we took advantage of the fact that the order of a Sylow p-subgroup always divides the dimension of a projective module. In order to obtain similar results for other relative projective homomorphisms, we must produce similar information in this case. This will be provided by Green's Theorem (11.10) which asserts that if an absolutely indecomposable module has vertex V, and $V \leq Q \in Syl_p(G)$, then $|Q : V|$ divides the dimension (or rank) of the module.

Next we recall that a basic property of the tensor product is that if $A, B \in \mathbf{M}_\Theta(G)$, where Θ is a principal ideal domain, and M is an Θ-pure submodule of A, then

$$(24) \qquad 0 \to M \otimes B \to A \otimes B \to A/M \otimes B \to 0$$

is exact. This observation and Schanuel's Lemma (cf. the opening remarks of Section 2) easily yield

Lemma 6.13. Let $A, B \in \mathbf{M}_\Theta(G)$, where Θ equals R or F. Then there exist projective modules $Q_1, Q_2, P_1, P_2 \in \mathbf{M}_\Theta(G)$ such that

$$(25) \qquad A \otimes B^* \oplus P_1 \simeq \Omega A \otimes (\Omega B)^* \oplus P_2$$

$$(26) \qquad A \otimes B \oplus Q_1 \simeq \Omega A \otimes \mho B \oplus Q_2 \quad .$$

Proof: By Lemma 1.10.5, (25) and (26) are equivalent. To prove (25), denote as usual the projective cover of M by P_M. Then

$$(27) \qquad 0 \to \Omega A \otimes (\Omega B)^* \to P_A \otimes (\Omega B)^* \to A \otimes (\Omega B)^* \to 0$$

$$(28) \qquad 0 \to A \otimes B^* \to A \otimes P_B^* \to A \otimes (\Omega B)^* \to 0$$

are exact sequences, and the middle terms are projective by Corollary 6.4. Thus (25) follows from (27) and (28) and Schanuel's Lemma.

Corollary 6.14 (Feit (1969)). Let $A, B \in \mathbf{M}_\Theta(G)$, and let

$\mathcal{H}$ be any family of subgroups of G. Then

(29) $$(A, B)^{\mathcal{H},G} \simeq (\Omega A, \Omega B)^{\mathcal{H},G} \quad .$$

In particular, if $\Theta = F$,

(30) $$\mathrm{Ext}^1_{F[G]}(A, B) \simeq \mathrm{Ext}^1_{F[G]}(\Omega A, \Omega B) \quad .$$

Proof: By Lemma 6.13,

(31) $$\begin{aligned}(I, A^* \otimes B)^{\mathcal{H},G} &\simeq (I, (A^* \otimes B) \oplus P_1)^{\mathcal{H},G} \\ &\simeq (I, ((\Omega A)^* \otimes \Omega B) \oplus P_2)^{\mathcal{H},G} \\ &\simeq (I, (\Omega A)^* \otimes \Omega B)^{\mathcal{H},G}\end{aligned}$$

for suitable projective modules P_1 and P_2, where I denotes the trivial $\Theta[G]$-module, as any map into a projective module is $\mathcal{H}$-projective by Theorem 5.5. Thus (29) follows from Theorem 6.10. Now (30) is obtained by choosing $\mathcal{H} = 1$.

Remark: There is a very natural isomorphism in (30), namely the following: Let $\phi \in (A, B)^G$. As P_A is projective, ϕ may be extended to $\tilde{\phi} \in (P_A, P_B)^G$. Now, the restriction ϕ' of $\tilde{\phi}$ to ΩA maps into ΩB and an easy argument shows that ϕ' uniquely determined modulo $(\Omega A, \Omega B)^G_1$ by ϕ, although of course ϕ' itself and even $\tilde{\phi}$ are not uniquely determined. We shall return to this in Proposition 6.17.

The following is well-known, too.

Corollary 6.15. Let $A, B, C \in \mathbf{M}_F(G)$. Then

(32) $$\mathrm{Ext}^1_{F[G]}(A \otimes B,\ C) \simeq \mathrm{Ext}^1_{F[G]}(A,\ B^* \otimes C) \quad .$$

<u>Proof</u>: By definition,

(33) $$\begin{aligned}\mathrm{Ext}^1_{F[G]}(A \otimes B,\ C) &= (\Omega(A \otimes B),\ C)^{1,G} \\ &= (A \otimes B,\ \mho C)^{1,G}\end{aligned}$$

by Corollary 6.14,

$$= (\Omega A \otimes \mho B,\ \mho C)^{1,G}$$

by Lemma 6.13

$$= (\Omega A,\ \Omega(B^*) \otimes \mho C)^{1,G}$$

by Theorem 6.10 and Lemma I.10.3

$$= (\Omega A,\ B^* \otimes C)^{1,G}$$

by Lemma 6.13

$$= \mathrm{Ext}^1_{F[G]}(A,\ B^* \otimes C)^{1,G}$$

by definition.

We end with a little known result of Gabriel (1980), which applied twice yields (29) in the case where $\mathcal{H} = 1$. However, we point out that in (34), 1 cannot be replaced by $\mathcal{H}$.

<u>Corollary 6.16</u>. Let $A,\ B \in \mathbf{M}_\Theta(G)$. Then

(34) $$(A,\ B)^{1,G} \simeq (B,\ \Omega A)^{1,G} \quad .$$

In particular, we observe that

i) $(A, A)^{1,G} \simeq (A, \Omega A)^{1,G} \simeq (\Omega A, \Omega A)^{1,G}$.

ii) $\mathrm{Ext}^1_{\Theta[G]}(B, \Omega^2 A) \simeq (A, B)^{1,G}$.

Proof: By Theorem 6.9,

$$(A, B)^{1,G} \simeq (A \otimes B^*, I)^{1,G} \simeq (I, \Omega(A \otimes B^*))^{1,G} \tag{35}$$

by Corollary 6.11,

$$\simeq (I, \Omega A \otimes B^*)^{1,G}$$

by Corollary 6.5 and Schanuel's Lemma

$$\simeq (B, \Omega A)^{1,G}$$

by Theorem 6.10. Now i) and ii) are immediate consequences.

Let $A \in \mathbf{M}_\Theta(G)$. By Corollary 6.14, the rings $(A, A)^{1,G}$ and $(\Omega A, \Omega A)^{1,G}$ are rings and isomorphic as Θ-modules. It seems natural to ask if they in fact are isomorphic as algebras. And indeed they are, which will prove very useful in Section 8.

Proposition 6.17. Let $A \in \mathbf{M}_\Theta(G)$. Then

$$(A, A)^{1,G} \simeq (\Omega A, \Omega A)^{1,G} \tag{36}$$

as rings.

Proof: Set $E_1 = (\Omega A, \Omega A)^G$ and $E_2 = (A, A)^G$, and denote E_i modulo the space of the projective maps by $\bar{E}_i$. Let $\phi \in E_1$. Then ϕ can be extended to $\Phi \in (P_A, P_A)^G$, as P_A is injective in the category of Θ-free $\Theta[G]$-modules. Furthermore, if Φ_1 and Φ_2 are both extensions, then $\Phi_1 - \Phi_2$ contains ΩA in the kernel. Thus $(\Phi_1 - \Phi_2) \in (A, P_A)^G$. Now, the composite of this map and the canonical map $P_A \longrightarrow A$ is a projective endomorphism of A, and consequently $\overline{\phi'} \in \overline{E}_2$ only depends on ϕ and therefore is well defined in $\overline{E}_2$. It follows that $\Lambda : E_1 \longrightarrow \overline{E}_2$ given by $\Lambda(\phi) = \overline{\phi'}$ is a ring homomorphism, as it is Θ-linear by construction. Assume next that ϕ is projective. Then ϕ may be factored through P_A, i.e., there exists an extension Φ of ϕ in $(P_A, P_A)^G$ such that $\Phi(P_A) \subseteq \Omega A$ which forces $\Lambda(\phi) = 0$. Thus Λ induces a ring homomorphism from $\overline{E}_1$ into $\overline{E}_2$. Next we prove that Λ is surjective: Let $\alpha \in \overline{E}_2$. Then α is induced from some $\Phi \in (P_A, P_A)^G$, as P_A is projective. Furthermore, by construction of Φ, $\Phi(\Omega A) \subseteq \Omega A$, and thus α may be obtained from $\phi \in E_1$, where ϕ is the restriction of Φ to ΩA, in the way described above, which establish the surjectivity. Finally, injectivity follows by duality: Set $B = (\Omega A)^*$. Then $\Omega B \simeq A^*$ by Lemma 1.10.6. But as $(A, A)^G \simeq ((A^*, A^*)^G)^{op}$, we thus get a surjective homomorphism from $\overline{E}_2$ onto $\overline{E}_1$. As Θ is a principal ideal domain and E_1 and E_2 have finite rank over Θ, isomorphism therefore follows.

We close this section with a few remarks on $M \otimes M$ and $M \otimes M^*$ for an $F[G]$-module M.

Consider first $M \otimes M$. Let $\{v_1, \ldots, v_n\}$ be a basis of M. Then $\{v_i \otimes v_j\}_{i,j}$ form a basis of $M \otimes M$. We now define an action of $\mathbf{Z}_2 = \langle \tau \rangle$ on $M \otimes M$, given by $(v_i \otimes v_j)\tau = v_j \otimes v_i$ and then extending linearly.

Obviously, G and $\mathbf{Z}_2$ commute in their action on $M \otimes M$. Thus any characteristic $\mathbf{Z}_2$-invariant subspace of $M \otimes M$ is an $F[G]$-

<u>Definition 6.18</u>. By the <u>exterior (2nd) power</u> of M, we mean the $F[G]$-submodule

$$(37) \qquad E_2(M) = \text{span}_F\{v_i \otimes v_j - v_j \otimes v_i\}$$

of dimension $\frac{n(n-1)}{2}$.

If char $F \neq 2$, $M \otimes M$ is a semisimple $F[\langle \tau \rangle]$-module, and we see that

$$(38) \qquad M \otimes M = E_2(M) \oplus S_2(M)$$

where

$$(39) \qquad S_2(M) = \text{span}_F\{v_i \otimes v_j + v_j \otimes v_i\} \quad .$$

If char $F = 2$, this is no longer true. However, in this case $M \otimes M = W_1 \oplus W_2$, where W_1 is a trivial $F[\langle \tau \rangle]$-module and W_2 is a projective $F[\langle \tau \rangle]$-module. Moreover, it is easy to see that as such, $\text{Soc}(W_2) = E_2(M)$. Also, the socle of $M \otimes M$ as an $F[\langle \tau \rangle]$-module, which is $W_1 \oplus E_2(M)$, is the $F[G]$-module

$$(40) \qquad E_2'(M) = \text{span}_F\{v_i \otimes v_i , v_i \otimes v_j\}$$

which is of dimension $\frac{n(n+1)}{2}$. Finally, we note that

$$(41) \qquad M/E_2'(M) \simeq E_2(M)$$

as $F[G]$-modules, and the isomorphism is induced from the linear map $v \longrightarrow v(1+\tau)$.

It is of course possible to consider $M^{(\otimes n)}$ for arbitrary n

and then let the symmetric group Σ_n on n letters act on this. Multiplying by various ideals of $F[\Sigma_n]$, we obtain various $F[G]$-submodules of $V^{\otimes n}$. This is described in detail in James (1980). As for applications this has played a major role in recent activities focussing on the determination of the simple modular representations of the sporadic simple groups. Naturally, only small values of n (≤ 5) are useful. One gets a good impression of the effectiveness in Thackeray (1981). The name of the game is always to start with a small dimensional simple module M which for some reason is known either through the very construction of the group or perhaps from the existence of a small dimensional representation in characteristic 0 of its covering group, and then consider $M \otimes M$. R. Parker has developed very ingenious methods by which it is possible by the help of a computer to decide whether or not a module is simple if its dimension is not too large. Using block theory, the subject of the next chapter, it is often possible to determine simple modules with very small vertices. We have already seen in Corollary I.16.8 how all simple modules which are projective may be determined from the character table, and we shall soon see that the simple module E of Corollary 16.11 has $\mathbf{Z}_2$ as vertex <u>and</u>, which is more important, a finite group G has such a simple module if and only if there exists an involution $\tau \in G$ such that $\overline{C} = C_G(\tau)/\langle \tau \rangle$ has a 2-block with exactly one irreducible character, a fact that is immediate from the character table of $\overline{C}$, as discussed in Ch. I, Sec. 16.

Finally, a few words on $M \otimes M^*$. We have already seen in Lemma 6.1 that $M \otimes M^* \simeq (M, M)$ as an $F[G]$-module. Also, it follows immediately from Corollary 6.9 that $M \otimes M^*$ has a submodule as well as a factor module isomorphic to the trivial $F[G]$-module I. We now have the following remarkable result.

Lemma 6.18. Let M be an $F[G]$-module. Then

i) Assume $\operatorname{char} F \nmid \dim_F M$. Then $I \mid M \otimes M^*$.

ii) The converse of i) holds for all modules M with $(M, M)^G \simeq F$.

Proof: i) (Feit (1982), p. 98). Set $m = \dim_F M$. Then $M \otimes M^* \simeq (M, M) \simeq \operatorname{Mat}_n(F) = W$ as $F[G]$-modules, where the action on $\operatorname{Mat}_n(F)$ of G is given by conjugation by definition of the G-action on (M, M). The subspace W_0 consisting of matrices of trace 0 is an $F[G]$-submodule of W of codimension 1, and $W/W_0 \simeq I$. Moreover the subspace W_1 consisting of scalar matrices is an $F[G]$-submodule isomorphic to I. Obviously, $W_0 \cap W_1 = 0$ if and only if $\dim F \nmid m$, and i) follows.

ii) Assume $(M, M)^G = F = W^G$. Then in fact $W^G = W_1$. Assume $\operatorname{char} F \mid \dim_F M$. Then $W_1 \subseteq W_0$. Now assume $W \simeq I \oplus V$. Since W_1 is the unique trivial submodule of W, $W/W_1 \simeq V$. However, as W is self dual, $(W, I)^G \simeq F$ and thus

$$(W/W_1, I)^G \simeq (V, I)^G = 0 \tag{42}$$

a contradiction as $W_1 \subseteq W_0$ and $W/W_1 \simeq I$.

Corollary 6.19. Let (F, R, S) be a p-modular system, and let S be a splitting field of $S[G]$. Then

i) Let E be a simple $F[G]$-module. Then $\dim_F(E)$ is prime to p if and only if I is a direct summand of $E \otimes E^*$.

ii) Let χ be an irreducible character of $S[G]$, and assume χ has an R-form M such that all endomorphisms of $\overline{M} = M/M\pi$ are liftable. Then $\chi(1)$ is prime to p if and only if I is a direct summand of $\overline{M} \otimes \overline{M}^*$.

Proof: i) is clear.

ii) As χ is irreducible, $(M, M)^G \simeq R$ and thus $(\overline{M}, \overline{M})^G \simeq F$ by assumption.

7. The Green ring.

Let G be a finite group and F any field of characteristic p.

Definition 7.1. By the Green ring $a(G) = a_F(G)$ we understand the free $\mathbf{Z}$-module spanned by the indecomposable elements of $\mathbf{M}_F(G)$. Thus addition in $a(G)$ corresponds to taking direct sums of indecomposable modules.

The multiplicative structure on $a(G)$ is imposed from tensor products of modules. If $M_1, M_2 \in \mathbf{M}_F(G)$ and the element of $a(G)$ corresponding to $M \in \mathbf{M}_F(G)$ is denoted by $a(M)$, we set

(1) $$a(M_1)\cdot a(M_2) := a(M_1 \otimes M_2) \quad .$$

This obviously makes $a(G)$ into a commutative ring. We now set

(2) $$A(G) = a_F(G) \otimes_{\mathbf{Z}} \mathbf{C}$$

which will be called the complex Green ring. In the following, we will simply denote the element of $A(G)$ corresponding to $M \in \mathbf{M}_F(G)$ by M.

Let $H \leq G$. We then have a ring homomorphism

(3) $$r_{G,H} : A(G) \longrightarrow A(H)$$

induced from restriction of modules (on which we will still use our notation $\downarrow H$ or $\downarrow^{H}_{G}$) and a linear map

$$i_{H,G} : A(H) \longrightarrow A(G) \tag{4}$$

induced from induction of modules (on which we likewise continue with $\uparrow G$ or $\uparrow^{G}_{H}$).

Denote the linear span in $A(G)$ of the direct summands of modules of the form $N\uparrow^{G}_{H}$, where $N \in \mathbf{M}_F(H)$, by $A(G, H)$. Then Corollary 6.3 states that

<u>Lemma 7.2.</u> $A(G, H)$ and $\operatorname{Im} i_{H,G} \subseteq A(G, H)$ are both ideals in $A(G)$.

For $\mathcal{H}$ any family of subgroups of G, we let $A(G, \mathcal{H})$ denote the ideal $\sum_{U \in \mathcal{H}} A(G, U)$. In particular, $A(G, 1) = K_0(G)$ is the ideal spanned by projective $F[G]$-modules. Moreover we set

$$A_{\mathcal{H}}(G) = \operatorname{span}_{\mathbf{C}} \{X - X' - X''\} \cap \left(\bigcap_{U \in \mathcal{H}} \operatorname{Ker} r_{G,U}\right) \tag{5}$$

where $0 \to X' \to X \to X'' \to 0$ runs through all short exact sequences.

<u>Lemma 7.3.</u> $A_{\mathcal{H}}(G)$ is an ideal in $A(G)$.

<u>Proof</u>: By definition, it is an intersection of ideals.

For completeness, we mention the following

<u>Theorem 7.4</u> (Dress (1974)). Let $\mathcal{H}$ be any family of subgroups of G. Then

$$A(G) = A(G, \mathcal{H}) \oplus A_{\mathcal{H}}(G) \quad . \tag{6}$$

Proof: We shall not give the full proof but only observe that $A(G, \mathcal{H})A_{\mathcal{H}}(G) = 0$ by Corollary 6.5. Thus it suffices to prove that $A(G) = A(G, \mathcal{H}) + A_{\mathcal{H}}(G)$. In the case where $H = 1$ this is an old and well-known result: Let $M \in \mathbf{M}_F(G)$ and let $M_1, \ldots, M_n$ denote all the simple composition factors of M. Then $M - \sum_{i=1}^{n} M_i \in A_1(G)$ by definition. Thus it suffices to prove that $M_i \in K_0(G)$ for all i. But this follows from Theorem I.15.9.

It was pointed out to the author by Irving Reiner that (6) can be derived from Dress (1974).

We shall be more interested in the following beautiful results, which were recently proved by D. Benson, using the theory of almost split sequences (see Benson and Parker (1983)). Our proof here will be completely elementary, though slightly tricky perhaps.

Theorem 7.5. Let $H \leq G$. Then

(7) $$A(G) = \operatorname{Im} i_{H,G} \oplus \operatorname{Ker} r_{G,H}$$

as a direct sum of ideals. Likewise,

(8) $$A(H) = \operatorname{Ker} i_{H,G} \oplus \operatorname{Im} r_{G,H}$$

as a direct sum of vector spaces.

Before we prove this, we state the following striking

Corollary 7.6. i) Let $N_1, N_2 \in \mathbf{M}_F(H)$ and assume $(N_1^{\uparrow G})_{\downarrow H} \simeq (N_2^{\uparrow G})_{\downarrow H}$. Then $N_1^{\uparrow G} \simeq N_2^{\uparrow G}$.

ii) Let $M_1, M_2 \in \mathbf{M}_F(G)$ and assume $(M_{1\downarrow H})^{\uparrow G} \simeq (M_{2\downarrow H})^{\uparrow G}$. Then $M_{1\downarrow H} \simeq M_{2\downarrow H}$.

Proof: Clear. Note that we only use the fact that $\operatorname{Im} i_{H,G} \cap \operatorname{Ker} r_{G,H} = \operatorname{Ker} i_{H,G} \cap \operatorname{Im} r_{G,H} = 0$.

Proof of Theorem 7.5. We first prove (7), using induction on $|H|$. If $|H| = 1$, the codimension of $\operatorname{Ker} r_{G,1}$ is 1, which also equals $\dim_F(\operatorname{Im} i_{1,G})$. As $i_{1,G}(1) \notin \operatorname{Ker} r_{G,1}$, this case is settled. Suppose therefore that $|H| > 1$, and set, for each $K \leq H$, $\mathfrak{A}_K = \operatorname{Im} i_{K,G}$ and $\mathfrak{L}_K = \operatorname{Ker} r_{G,K}$. Then $\mathfrak{A}_K$ and $\mathfrak{L}_K$ are both ideals in $A(G)$. Moreover, $\mathfrak{A}_K \mathfrak{L}_K = 0$ by Corollary 6.3. By induction, $A(G) = \mathfrak{A}_K \oplus \mathfrak{L}_K$ for any $K < H$. As this is an ideal decomposition, we therefore obtain that

(9) $$A(G) = \sum_{K<H} \mathfrak{A}_K + \bigcap_{K<H} \mathfrak{L}_K$$

and thus

(10) $$\operatorname{Im} r_{G,H} = r_{G,H}\Big(\sum_{K<H} \mathfrak{A}_K\Big) + \bigcap_{K<H} \operatorname{Ker} r_{H,K} .$$

Let $1 = a+b$ in this decomposition. In particular, b is $N_G(H)$-invariant, and by Mackey decomposition we get $r_{G,H}(i_{H,G}(b)) = |N_G(H) : H| b$. Thus $b \in r_{G,H}(\mathfrak{A}_H)$, and consequently $\operatorname{Im} r_{G,H} = r_{G,H}(\mathfrak{A}_H)$, which proves that $A(G) = \mathfrak{A}_H + \mathfrak{L}_H$ as obviously $\mathfrak{A}_K \subseteq \mathfrak{A}_H$ for $K \leq H$. Now, as $\mathfrak{A}_H \mathfrak{L}_H = 0$, this sum is direct.

To prove (8), set $V = \operatorname{Ker} i_{H,G}$, $W = \operatorname{Im} r_{G,H}$. In order to show that $A(H) = V + W$, it suffices to prove that $M \in V + W$ for any $M \in \mathbf{M}_F(H)$. To prove this we use induction on $|K|$, where $K \leq H$, to show that $(M_{\downarrow K})^{\uparrow H} \in V + W$. The case $K = 1$ is trivial, as obviously $F[H] = \frac{1}{|G:H|} r_{GH}(F[G])$ as elements in the Green ring. Now, we furthermore have that

$$(11) \qquad ((M_{\downarrow K})^{\uparrow G})_{\downarrow H} = \sum_{\gamma \in K \backslash G/H} (M \otimes \gamma)\uparrow^{H}_{K^{\gamma} \cap H}$$

$$\equiv m(M_{\downarrow K})^{\uparrow H} + \sum_{|K^{\gamma} \cap H| < |K|} (M \otimes \gamma)\uparrow^{H}_{K^{\gamma} \cap H} \mod V$$

as elements in $A(H)$, where m is the cardinality of the set $\{\gamma \in K \backslash G/H \mid |K^{\gamma} \cap H| = |K|\}$. Observe that $(M_{\downarrow K})^{\uparrow H} - ((M \otimes \gamma)_{\downarrow K^{\gamma}})^{\uparrow H} \in V$ for all γ in this set. As the left-hand side of (11) lies in W, we are done by induction.

It remains to prove that $V \cap W = 0$. Let $x \in V \cap W$, and let $u \in \operatorname{Im} i_{H,G}$ with $x = r_{G,H}(u)$, using the first part of the theorem. We claim that $xA(H) \subseteq V$. Indeed, if $y \in A(G)$, then

$$(12) \qquad i_{H,G}(xr_{G,H}(y)) = i_{H,G}(x)y = 0$$

by Corollary 6.3. If $z \in V$, then

$$(13) \qquad i_{H,G}(xz) = i_{H,G}(r_{G,H}(u)z) = ui_{H,G}(z) = 0$$

as well. As $A(H) = V + W$, the claim follows. Finally, let $s \in A(H)$. Then $ui_{H,G}(s) = i_{H,G}(xs) = 0$. Thus $u\mathfrak{A}_H = 0$. As $u \in \mathfrak{A}_H$, we also have that $u\mathfrak{L}_H = 0$. Hence $u = 0$ by (7), and consequently $x = 0$.

A number of our results in earler sections may of course be expressed as properties of the Green ring, but we shall refrain from actually doing that, and just refer the reader for this, as well as several further results on the Green ring to Feit (1982). Instead, we change our focus of interest somewhat and introduce the idea of periodicity.

Again, let Θ equal F or R, where (F, R, S) is a p-modular system. Recall that $M \in \mathbf{M}_\Theta(G)$ is called periodic if there exists an exact sequence

$$(14) \qquad 0 \to M \to P_n \to P_{n-1} \to \dots \to P_1 \to M \to 0$$

with P_i projective for $i = 1, \dots, n$.

A classical result states the following

Proposition 7.7. Let P be an elementary abelian p-group. Then the trivial 1-dimensional $\mathbb{C}[P]$-module is periodic if and only if P is cyclic.

Proof: See Cartan & Eilenberg (1956), Ch. XII, Section 11.

Corollary 7.8. A(G) is finite dimensional if and only if a Sylow p-subgroup of G is cyclic.

Proof: We have already seen that a cyclic p-group only has finitely many indecomposable non-isomorphic modules. Therefore, if $Q \in \mathrm{Syl}_p(G)$ is cyclic, Corollary 2.5 implies that F[G] only has finitely many non-isomorphic modules. If Q is not cyclic, Q has a factor group isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$, and by Proposition 7.7, the family $\{\Omega^i I\}_i$, where I is the trivial $F[\mathbf{Z}_p \oplus \mathbf{Z}_p]$-module, contains infinitely many non-isomorphic indecomposable modules, as obviously I is periodic if and only if $\Omega^i I$ is for all i. Set $N_i = \Omega^i I$, considered as an F[Q]-module by inflation. As $N_i | (N_i^{\uparrow G})_{\downarrow Q}$ it follows that $\{N_i^{\uparrow G}\}$ must contain infinitely many non-isomorphic indecomposable direct summands. We also get, obviously, by the same reasoning

Corollary 7.9. Let $H \leq G$ such that a Sylow p-subgroup of

H is properly contained in one of G. Then

i) $A(G)/A(G, H)$ is finite dimensional if and only if a Sylow p-subgroup of G is cyclic.

ii) $A(G, H)$ is finitely dimensional if and only if a Sylow p-subgroup of H is cyclic.

Remark: Let Q be a p-group. It then follows from Schanuel's Lemma that if the trivial $F[Q]$-module is periodic then so is that of every proper subgroup of Q. Thus it follows from Proposition 7.7 that if the trivial $F[Q]$-module is periodic, then the p-rank of Q is 1, in other words Q is cyclic or, if $p = 2$, quaternion. The converse immediately follows from the example of Section I, 8 if Q is cyclic. We mention without proof that the converse holds as well if Q is quaternion. For a proof see Cartan-Eilenberg (1956), Ch. XII, Section 11.

The key to our observations above was periodicity. Let us proceed to discuss another important idea, complexity, introduced in Alperin (1977). Here we follow Alperin & Evens (1981), though. Let M be an $F[G]$-module and let

(15) $$\cdots \longrightarrow P_2 \xrightarrow{\phi_3} P_1 \xrightarrow{\phi_2} P_0 \xrightarrow{\phi_1} M \xrightarrow{\phi_0} 0$$

be the minimal resolution of M, i.e., P_i is the projective cover of $\operatorname{Ker} \phi_i$ and $\phi_{i+1} : P_i \longrightarrow P_{i-1}$ is a surjection.

Definition 7.10. The complexity $c = c_G(M)$ of M is the smallest nonnegative integer such that there exists a positive number λ with

(16) $$\dim_F P_n \leq \lambda n^{c-1}$$

for all n sufficiently large.

It has been shown in Lewis (1968) that if Q is a p-group and I is the trivial module, then $c_Q(I)$ exists and is bounded by a, where $|Q| = p^a$. If now $Q \in \mathrm{Syl}_p(G)$ we may induce any projective $F[Q]$ resolution to $F[G]$ to get a projective $F[G]$ resolution of I, which shows that $c_G(I) \leq c_Q(I)$. However, this resolution tensored with M will provide a projective resolution of M and therefore $c_G(M) = c$ exists and is less than or equal to $c_G(I)$. Thus the complexity of a module always exists.

It follows that

i) The complexity of M is 0 if and only if M is projective.

ii) The complexity of M is 1 if and only if M is periodic. (One way is trivial, the other not.)

Now the main result of Alperin & Evens (1981), which we shall quote without proof, is

Theorem 7.11. With the notation above,

$$c_G(M) = \max_E \{c_E(M_{\downarrow E})\} \tag{17}$$

where E runs through the set of elementary abelian p-subgroups of G.

This has a number of interesting consequences, of which we mention

Corollary 7.12 (Chouinard (1976). M is projective if and only if $M_{\downarrow E}$ is projective for every elementary abelian p-subgroup E of G.

Proof: Obvious (by i)).

Corollary 7.13. M is periodic if and only if $M_{\downarrow E}$ is periodic for every elementary abelian p-subgroup E of G.

Proof: Obvious (by ii)).

Corollary 7.13. Let M be indecomposable and let V be a vertex of M. Then the complexity of M is at most the p-rank of V.

Proof: By the result of Lewis (1968) referred to above.

This whole theory has been further developed and many other aspects have emerged. We refer to Carlson (1977), (1978), (1979), Kroll (1980), Avrunin (1981) and Avrunin & Scott (1982a), (1982b).

8. Endomorphism rings.

We continue to let F be an arbitrary field.

Let A, B $\in \mathbf{M}_F(G)$ and set $E(A) = (A, A)^G$ and $\overline{E}(A) = (A, A)^{1,G}$ (recall that $(A, A)_1^G$ is an ideal in $(A, A)^G$). We begin by stating the following well-known observations, where always $a \in A$, $\alpha \in E(A)$, $b \in B$, $\beta \in E(B)$.

i) A is a left E(A)-module by $\alpha a := \alpha(a)$.

ii) A^* is a right E(A)-module by $a^*\alpha := a^* \circ \alpha$.

iii) (A, B) is an $E(B) \times E(A)$-bimodule by $\beta\psi\alpha := \beta \circ \psi \circ \alpha$ for $\psi \in (A, B)$.

iv) $B \otimes A^*$ is an $E(B) \times E(A)$-bimodule by $\beta(b \otimes a^*)\alpha := \beta b \otimes a^*\alpha$. Likewise, $A^* \otimes B$ is an $E(B) \times E(A)$-bimodule by $\beta(a^* \otimes b)\alpha := a^*\alpha \otimes \beta b$. As such they are isomorphic.

v) $(B^* \otimes A, F)$ is an $E(B) \times E(A)$-bimodule by

$\beta\phi\alpha(b^* \otimes a) = \phi(b^*\beta \otimes \alpha a)$ for $\phi \in (B^* \otimes A, F)$, where F is considered as the trivial $F[G]$-module.

The following supplement to Lemma 6.1 is also well-known, but we include a proof for completeness.

Proposition 8.1. As $E(B) \times E(A)$-bimodules,

(1) $$B \otimes A^* \simeq (A, B) \simeq (B^* \otimes A, F) \quad .$$

Proof: First isomorphism: We just have to check that Δ defined in 6, (1) is an isomorphism. Recall that $\Delta(b \otimes a^*) = \phi_{b,a^*}$, where $\phi_{b,a^*}(x) = a^*(x)b$ for $x \in A$. Hence

(21) $$\begin{aligned} \Delta(\beta(b \otimes a^*)\alpha) &= \phi_{\beta b, a^*\alpha} \\ &= \beta \circ \phi_{b,a^*} \circ \alpha \\ &= \beta\Delta(b \otimes a^*)\alpha \end{aligned}$$

as

(3) $$\phi_{\beta b, a^*\alpha}(x) = a^*(\alpha(x))\beta(b) = \beta(a^*(\alpha(x))b) \quad .$$

Second isomorphism: Again we recall that $\nabla : (A, B) \longrightarrow (B^* \otimes A, F)$ given by

(4) $$[\nabla(\psi)](b^* \otimes a) = b^*(\psi(a))$$

for $\psi \in (A, B)$ is an $F[G]$-isomorphism. Now

$$(5)\qquad [\nabla(\beta\psi\alpha)](b^* \otimes a) = b^*(\beta \circ \psi \circ \alpha(a))$$
$$= [\nabla(\psi)](b^*\beta \otimes \alpha a)$$
$$= [\beta\nabla(\psi)\alpha](b^* \otimes a) \quad .$$

<u>Corollary 8.2</u>. As $\overline{E}(B) \times \overline{E}(A)$-bimodules

$$(6)\qquad (B \otimes A^*)^{1,G} \simeq (A, B)^{1,G} \simeq (B^* \otimes A, F)^{1,G} \quad .$$

<u>Proof</u>: By Theorem 6.10.

This enables us to improve Corollary 6.16 by taking advantage of the fact that $\overline{E}(A) \simeq \overline{E}(\Omega A)$ as rings (Proposition 6.17).

Define

$$(7)\qquad \Gamma : ((A \otimes B^*)^*)^G \longrightarrow ((\Omega A \otimes B^*)^{1,G})^*$$

as follows: Let P_A be the projective cover of A, and set $V = \Omega A \otimes B^* \subseteq P_A \otimes B^*$. We may then identify $A \otimes B^*$ with $P_A \otimes B^*/V$. Now, for $x \in P_A$, $y \in B^*$ and $\phi \in ((A \otimes B^*)^*)^G$, set

$$(8)\qquad [\Gamma(\phi)]((x \otimes y^*)(\sum_{g\in G} g) + V_1^G) = \phi(x \otimes y^* + V) \quad .$$

Observe that every element of $V^G \subseteq (P_M \otimes N^*)^G$ is of the form $(x \otimes y^*)(\sum_{g\in G} g)$, as $P_A \otimes B^*$ is projective. To see that Γ is well defined, let $a_1, a_2 \in P_A \otimes B^*$ and assume $(a_1-a_2)(\sum_{g\in G} g) \in V_1^G$. Then there exists $a \in V$ such that $(a-(a_1-a_2))(\sum_{g\in G} g) = 0$ by definition of V_1^G, which means that $a-(a_1-a_2) \in \operatorname{Ker} \phi$, as ϕ maps onto the trivial $F[G]$-module. Thus $\phi(a_1-a_2+V) = \phi(a+V) = 0$.

Assume finally that $\phi \in ((A \otimes B^*)^*)_1^G$, say $\phi = \lambda(\sum_{g\in G} g)$

where $\lambda \in (A \otimes B^*)^*$. Then

$$(9) \qquad \phi(x \otimes y^* + V) = \lambda(x \otimes y^*(\sum_{g \in G} g) + V) = 0$$

as $(x \otimes y)(\sum_{g \in G} g) = 0$. In fact we have

<u>Proposition 8.3</u>. The map Γ of (8) induces an $\overline{E}(B) \times \overline{E}(A)$-bimodule isomorphism

$$(10) \qquad ((A \otimes B^*)^*)^{1,G} \simeq ((\Omega A \otimes B^*)^{1,G})^* \quad .$$

<u>Proof</u>: Γ is obviously a surjective $\overline{E}(B) \times \overline{E}(A)$-bimodule homomorphism by definition. As the two modules have the same dimension by Corollaries 6.16 and 8.2, Γ is therefore an isomorphism.

We may now prove (see also Gabriel (1980) and Auslander & Reiten (1975)),

<u>Theorem 8.4</u>. i) There is a natural $\overline{E}(B) \times \overline{E}(A)$-bimodule isomorphism

$$(11) \qquad (A, B)^{1,G} \simeq ((B, \Omega A)^{1,G})^*$$

where the action on the right is given through Corollary 8.2 and the isomorphism $\overline{E}(A) \simeq \overline{E}(\Omega A)$ of Proposition 6.17.

ii) There is a natural $\overline{E}(B) \times \overline{E}(A)$-bimodule isomorphism

$$(12) \qquad (A, B)^{1,G} \simeq \mathrm{Ext}^1_G(B, \Omega^2 A)^* \quad .$$

<u>Proof</u>: i) By Corollary 8.2, $(A, B)^{1,G} \simeq ((A \otimes B^*)^*)^{1,G}$ as $\overline{E}(B) \times \overline{E}(A)$-bimodules, and $((B, \Omega A)^{1,G})^* \simeq ((\Omega A \otimes B^*)^{1,G})^*$ as

$\bar{E}(B) \times \bar{E}(\Omega A)$-bimodules. Thus Proposition 8.3 reduces the proof of i) to showing that if $\alpha \in E(P_A)$ and $\bar{\alpha} \longrightarrow \bar{\alpha}'$ is the isomorphism of Proposition 6.17, then $\psi\bar{\alpha} = \psi\bar{\alpha}'$ for all $\psi \in ((\Omega A \otimes B^*)^{1,G})^*$. To see this, let $\phi \in ((A \otimes B^*)^*)^G$ with $\psi = \Gamma(\phi)$ in the notation of Proposition 8.3. Then

$$(13) \qquad [\psi\bar{\alpha}]((x \otimes y^*)(\sum_{g \in G} g) + V_1^G) = \phi(\alpha(x) \otimes y^* + V)$$

by (8), while

$$(14) \qquad \begin{aligned} [\psi\bar{\alpha}']((x \otimes y^*)(\sum_{g \in G} g) + V_1^G) \\ = \psi((\alpha(x) \otimes y^*)(\sum_{g \in G} g) + V_1^G) \\ = \phi(\alpha(x) \otimes y^* + V) \end{aligned}$$

by definition of $\psi\bar{\alpha}'$ and Proposition 8.3.

ii) follows by applying i) thrice (so it must be true! See Carroll (1875)), as

$$(15) \qquad \mathrm{Ext}_G^1(B, \Omega^2 A) = (\Omega B, \Omega^2 A)^{1,G} .$$

9. Almost split sequences.

We proceed to discuss an important application of Theorem 8.4, namely the existence of almost split sequences, or Auslander-Reiten sequences. The original theory was developed in Auslander & Reiten (1975) and dealt with arbitrary artinian algebras. Naturally, restricting our attention to finite group algebras, the task becomes considerably easier; in fact, we are quite close already. Our primary motiviation is a striking application of this to the complex Green ring, which has recently been observed in Benson & Parker (1983) and will be discussed in the following section.

Definition 9.1 (Auslander-Reiten). Let A be a finite dimensional algebra over the field F. An almost split sequence is a short exact non-split sequence of A-modules

$$(1) \qquad 0 \longrightarrow X \longrightarrow Y \xrightarrow{\pi} Z \longrightarrow 0$$

where X and Z are both indecomposable such that if W is any A-module and $\rho \in (W, Z)^A$ is not a split epimorphism, or in other words, unless $W \simeq \operatorname{Ker} \rho \oplus \operatorname{Im} \rho$ and $\operatorname{Im} \rho \simeq Z$, then ρ may be factored through π. We say that (1) terminates in Z.

The point of this of course is that whenever W is indecomposable and not isomorphic to Z, we obtain an induced short exact sequence

$$(2) \qquad 0 \to (W, X)^A \to (W, Y)^A \to (W, Z)^A \to 0$$

which will prove very useful.

Theorem 9.2 (Auslander-Reiten). Let G be a finite group and let F be a field of characteristic p, where p divides the order of G.

Let A be an indecomposable F[G]-module. Then there exists an almost split sequence terminating in A. Moreover this sequence is unique up to isomorphism, and its first term is isomorphic to $\Omega^2 A$. •

Proof: i) Existence. By Theorem 8.4,

$$(3) \qquad (\mathrm{Ext}^1_G(A,\ \Omega^2 A))^* \simeq (A,\ A)^{1,G}$$

as an $\bar{E}(A) \times \bar{E}(A)$-bimodule, with the notation of the previous section. As A is indecomposable, $\bar{E}(A)$ is a local ring by Lemma 1.5.3. Thus it follows from (3) that $\mathrm{Ext}^1_G(A,\ \Omega^2 A)$ has a unique minimal submodule as an $\bar{E}(A) \times \bar{E}(A)$-bimodule or even just as an $\bar{E}(A)$-module. Let $\gamma \in (\Omega A,\ \Omega^2 A)^G$ such that $\bar{\gamma} = \gamma + (\Omega A,\ \Omega^2 A)^G_1$ is a generator of this submodule. Let

$$(4) \qquad 0 \longrightarrow \Omega^2 A \xrightarrow{\alpha} X_A \xrightarrow{\mu} A \longrightarrow 0$$

be the uniquely determined non-split extension determined by γ (see appendix I). Thus X_A is the pushout of γ and the embedding $\Omega A \longrightarrow P_A$. Let $\Gamma : P_A \longrightarrow X_A$ be a homomorphism such that $\mu \circ \Gamma : P_A \longrightarrow A$ is the natural homomorphism. Thus we have the following commutative diagram of short exact sequences for B an arbitrary module and $f \in (B,\ A)^G$:

$$(5) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \Omega B & \xrightarrow{\beta} & P_B & \xrightarrow{\nu} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Omega A & \longrightarrow & P_A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \Omega^2 A & \xrightarrow{\alpha} & X_A & \xrightarrow{\mu} & A & \longrightarrow & 0
\end{array}$$

where $\hat{f}$ is an arbitrary lift of f and f' is the restriction to ΩB, as seen before. Now the following successive steps are evidently equivalent:

a) f is not a split epimorphism;

b) the induced map $f_* : (A,\ B)^G \longrightarrow (A,\ A)^G$ is not surjective;

c) the induced map $\overline{f}_*(A, B)^{1,G} \longrightarrow (A, A)^{1,G}$ is not surjective;

d) the induced map $\overline{f}^* : \mathrm{Ext}^1_G(A, \Omega^2 A) \longrightarrow \mathrm{Ext}^1_G(B, \Omega^2 A)$ obtained through Theorem 8.4 ii) is not injective;

e) $\overline{f}^*$ has $\overline{\gamma}$ in its kernel.

From now on the arguments are standard in homological algebra: That $\overline{f}^*(\overline{\gamma}) = 0$ means that $\gamma \circ f' \in (\Omega B, \Omega^2 A)^G_1$ by Theorem 8.4 i) and thus factors through the injective hull P_B of ΩB by Lemma 2.7, say through $\hat{\rho} : P_B \longrightarrow \Omega^2 A$. Hence

(6) $$[\Gamma \circ \hat{f} - \alpha \circ \hat{\rho}](\varepsilon(\Omega B)) = 0$$

and thus $\mathrm{Id} \circ f$ factors through $\rho \in (B, X_A)^G$ defined by $\rho(b) = [\Gamma \circ \hat{f} - \alpha \circ \hat{\rho}](\hat{b})$ for $b \in B$ arbitrary and $\hat{b} \in P_B$ such that $\nu(\hat{b}) = b$, and μ. As $\Omega^2 A$ is indecomposable by Lemma 10.6 v), i) therefore holds.

ii) Uniqueness. Suppose

(7) $$0 \to X_i \to Y_i \to A \to 0$$

are almost split sequences for $i = 1, 2$. Then we have the following commutative diagram

(8) $$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \rho_1} & & \| & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & Y_2 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \rho_2} & & \| & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where ρ_i exists by i) and σ_i is the restriction of ρ_i, for $i = 1, 2$. Consequently, if $\lambda = \sigma_2 \circ \sigma_1 \in (X, X)^G$ were nilpotent, (7) would split for $i = 1$, a contradiction. But then λ is in fact an isomorphism by Lemma 1.5.3. By symmetry it therefore follows that σ_1 is an isomorphism. Thus ρ_1 is an isomorphism by the Five Lemma (see Cartan & Eilenberg (1956)) and we are done.

Corollary 9.3. Let A be an indecomposable $F[G]$-module, and let

$$0 \to \Omega^2 A \to X_A \to A \to 0 \tag{9}$$

be the almost split sequence terminating in A. Let B be any indecomposable $F[G]$-module, not isomorphic to A. Then

$$0 \to (B, \Omega^2 A)^G \to (B, X_A)^G \to (B, A)^G \to 0 \tag{10}$$

is exact as well, while

$$0 \to (A, \Omega^2 A)^G \to (A, X_A)^G \to (A, A)^G \to \mathrm{Soc}(\mathrm{Ext}^1_G(A, \Omega^2 A)) \to 0 \tag{11}$$

is exact, where the last term is considered as an $E(A)$-module and (11) is the truncation of the long exact Ext sequence (see appendix I).

Proof: (10) has already been pointed out and is obvious anyway. To prove (11) we first observe that $(X_A, A)^G$ maps into a proper $E(A)$-submodule of $(A, A)^G = E(A)$, as the identity of A does not factor through X_A. Hence $(X_A, A)^G$ maps into $J((A, A)^G)$. Thus we in fact have a short exact sequence

$$0 \to (A, \Omega^2 A)^G \to (X_A, A)^G \to J((A, A)^G) \to 0 \tag{12}$$

as any element in $J((A, A)^G)$ is nilpotent and consequently factors through X_A. However, (11) and (12) are equivalent by Theorem 8.4 ii), as $E(A)/J(E(A)) \simeq \bar{E}(A)/J(\bar{E}(A))$ by Corollary 2.8.

Corollary 9.4. Same notation as above. Let $H \leq G$. Then A is H-projective if and only if (9) restricted to H does not split.

Proof: If A is H-projective, the restriction of (9) to H does not split by Corollary 2.4 v). Conversely, if M is not H-projective, the map $\varepsilon : (M_{\downarrow H})^{\uparrow G} \longrightarrow M$ of Sec. 2, (6) does not split and thus may be factored through X_M. However, ε considered as an F[H]-homomorphism does split, and consequently (9) restricted to H splits.

10. Inner products on the Green ring.

We continue with the notation of the previous sections. Most of the following observations are inspired by Benson & Parker (1983).

Definition 10.1. Let $A, B \in \mathbf{M}_F(G)$. We then set

(1) $$\langle A, B \rangle := \dim_F (A, B)^G$$

(2) $$\langle A, B \rangle_1 := \dim_F (A, B)_1^G$$

and extend these to be defined on the complex Green ring A(G) in the following way.

If $x = \sum \lambda_i M_i \in A(G)$, where $\lambda_i \in \mathbf{C}$ and $\{M_i\}$ is a set of representatives of the indecomposable modules in $\mathbf{M}_F(G)$, we set $x^* = \sum \bar{\lambda}_i M_i^*$. Now, as $(A, B)^G = (A^* \otimes B)^G$ we may extend $\langle ., . \rangle$ and $\langle ., . \rangle_1$ to maps $A(G) \times A(G) \longrightarrow \mathbf{C}$, linear in the second variable, antilinear in the first, and $\langle x, y \rangle$ (resp. $\langle x, y \rangle_1$) equals $\langle y^*, x^* \rangle$ (resp. $\langle y^*, x^* \rangle_1$). Also $(x^*)^* = x$.

The trivial $F[G]$-module $I = I_G$ will also be denoted by 1 as an element of $A(G)$. We now set

(3) $$u = u_G = P_I - \mho I, \quad v = v_G = P_I - \Omega I \quad .$$

The reason for dropping the subscript G is that by Schanuel's Lemma,

(4) $$u_{G,H}(u_G) = u_H, \quad r_{G,H}(v_G) = v_H \quad .$$

We now have

Lemma 10.2. With the above notation,

i) $M = u(P_M - \Omega M) = v(I_M - \mho M)$

ii) $uv = vu = 1$, and $u^* = v$

iii) $uM = I_M - \mho M$, and $vM = P_M - \Omega M$ for any $M \in \mathbf{M}_F(G)$.

Proof: By Lemma 6.13,

(5) $$\mho I \otimes \Omega M \simeq M \oplus P$$

where, as an element of $A(G)$, $P = P_I \Omega M - P_M$. As moreover $P_M = P_I P_M - \mho I P_M$ by Corollary 6.4, we consequently have

(6) $$\begin{aligned} u(P_M - \Omega M) &= P_I P_M - \mho I P_M - P_I \Omega M + \mho I \Omega M \\ &= P_M - P_I \Omega M + M + P_I \Omega M - P_M \\ &= M \quad . \end{aligned}$$

The second equality of i) is proved in the same way. Now ii) is a special case $(M = I)$ (that $u^* = v$ follows directly from the definition) and iii) follows from i) and ii).

<u>Lemma 10.3</u>. Let $x, y, z \in A(G)$. Then

(7) $$\langle xy, z \rangle = \langle x, y^* z \rangle \ ; \quad \langle xy, z \rangle_1 = \langle x, y^* z \rangle_1 \ .$$

Moreover,

(8) $$\langle x, y \rangle_1 = \langle ux, y \rangle \ ; \quad \langle x, y \rangle = \langle vx, y \rangle_1 \ .$$

<u>Proof</u>: (7) is an immediate consequence of the definition and Theorem 6.10. Also, Corollary 6.11 yields that

(9) $$\langle I, M \rangle_1 = \langle P_I, M \rangle - \langle \tilde{\Omega} I, M \rangle = \langle u, M \rangle$$

and hence $\langle 1, x \rangle_1 = \langle u, x \rangle$ for all $x \in A(G)$. Now the first part of (8) follows from (7) and the second part in turn by using that $uv = 1$.

Furthermore, as a corollary of Corollary 6.11, and by definition, we have

<u>Lemma 10.4</u>. Let $A, B \in \mathbf{M}_F(G)$. Then

i) $\langle A, B \rangle_1 \leq \langle A, B \rangle$.

ii) $<., .>_1$ is an inner product on $A(G)$, i.e., is symmetric.

iii) Assume $\langle A, B \rangle_1 \neq 0$. Then $\langle B, A \rangle \neq 0$.

<u>Proof</u>: i) is by definition.

ii) By Corollary 6.11, $\langle A, B \rangle_1$ equals the multiplicity of P_I as a direct summand of $A^* \otimes B$. As P_I is self-dual, this also equals that of P_I as a direct summand of $A \otimes B^*$.

iii) follows from i) and ii).

Remark: Observe that Benson & Parker (1983) defines $\langle .,.\rangle_1$ (which they denote $\langle .,.\rangle$) as $\left\langle M_i, M_j\right\rangle_1$ equal to the multiplicity of P_I as a direct summand of $M_i^* \otimes M_j$, rather than our (2), which are the same by Corollary 6.11.

We may now prove one of the main results of Benson & Parker (1983). For M a module, we denote the radical by $J(M)$.

Now, for each i, let

$$(10) \qquad 0 \to \Omega^2 M_i \to X_i \to M_i \to 0$$

be the almost split sequence terminating in M_i,

$$(11) \qquad 0 \to \Omega M_i \to Y_i \to \mho M_i \to 0$$

that terminating in $\mho M_i$. Set

$$(12) \qquad x_i = \begin{cases} M_i - J(M_i) & \text{if } M_i \text{ is projective} \\ M_i + \Omega^2 M_i - X_i & \text{otherwise} \end{cases} .$$

Similarly, we set

$$(13) \qquad y_i = \begin{cases} \mathrm{Soc}(M_i) & \text{if } M_i \text{ is projective} \\ Y_i - \mho M_i - \Omega M_i & \text{otherwise} \end{cases} .$$

Theorem 10.5. With the notation above, we have

$$(14) \qquad \left\langle M_i, x_j\right\rangle = \left\langle M_i, y_j\right\rangle_1 = \delta_{ij} .$$

In particular, $<.,.>$ and $<.,.>_1$ are non-singular in the sense that if there exists $a \in A(G)$ such that $\langle a, x\rangle_1$ (resp. $\langle a, x\rangle$ resp. $\langle x, a\rangle$) equals 0 for all x, then $a = 0$.

Proof: If M_j is not projective, $\langle M_i, x_j\rangle = \delta_{ij}$ by Corollary 9.3. If M_j is projective,

$$(15) \qquad \dim_F((M_i, M_j)^G) = \dim_F((M_i, J(M_j))^G) + \delta_{ij}$$

which is equivalent to the claim.

Likewise, $\langle M_i, y_j\rangle_1 = \delta_{ij}$ if M_j is projective. If not,

$$(16) \qquad \langle M_i, y_j\rangle_1 = \langle uM_i, y_j\rangle \quad .$$

by Lemma 10.3, (8).

$$= \langle I_{M_i} - \mho M_i, y_j\rangle$$

by Lemma 10.2 iii)

$$= \langle -\mho M_i, y_j\rangle$$

as I_{M_i} is projective

$$= \delta_{ij}$$

by the first part.

11. Induction from normal subgroups.

Let G be a finite group, and let H be a normal subgroup. Let Θ be a commutative ring such that Krull-Schmidt holds for

$\Theta[L]$-modules for any $L \leq G$. The purpose of this section is to discuss the decomposition of $N^{\uparrow G}$ for $N \in \mathbf{M}_{\Theta}(H)$ and related topics.

Definition 11.1. Same notation as above. If N is indecomposable, we define the inertial group T of N in G as

(1) $$T = T(N) = \{g \in G \mid N \simeq N \otimes g\} \quad .$$

If $T = G$, then N is said to be G-stable.

Let us start with the classical result.

Theorem 11.1 (Clifford (1937)). Let G be a finite group, $H \trianglelefteq G$ and let K be any field. Let M be a simple $K[G]$-module, and let N be a simple $K[H]$-submodule of M. The inertial group of N in G is denoted by T. Set $\tilde{N} = \{\Sigma Nt \mid t \in T\}$. Then

i) $M_{\downarrow H} \simeq \bigoplus_i N_i$, where for all i there exists $g_i \in G$ with $N_i \simeq Ng_i$ as $K[H]$-modules. In particular, $M_{\downarrow H}$ is semisimple.

ii) $M \simeq \tilde{N}{\uparrow}_T^G$.

iii) $M_{\downarrow H} \simeq \left(\bigoplus_{g \in T \backslash G} N \otimes g\right)^{(e)}$ where $\tilde{N} \simeq N^{(e)}$.

Proof: i) Obviously, $\sum_{g \in G} Ng$ is a $K[G]$-submodule of M, hence equal to M as M is simple. Also, Ng is simple for all $g \in G$, as N is. Thus M is a sum of simple $K[H]$-modules, and i) follows.

ii) By definition, $\tilde{N}$ is a sum of $K[H]$-modules isomorphic to N. Hence there exists $e \in \mathbb{N}$ such that $\tilde{N} \simeq N^{(e)}$, as N is simple. Moreover,

(2) $$M = \sum_{g \in G} Ng = \sum_{g_i \in T \backslash G} \tilde{N}g_i \quad .$$

However, as no direct summand of $\tilde{N}g_i$ is isomorphic to a direct summand of $\hat{N}g_j$ for $i \neq j$, this sum is a fortiori direct and ii) follows.

iii) is an immediate consequence of ii) and the fact that $\tilde{N} \simeq N^{(e)}$.

Theorem 11.2. Let F be an arbitrary field of characteristic p, and assume $H \trianglelefteq G$ is of index prime to p. Let $H \leq X \leq G$. Then

i) Let M be a simple $F[G]$-module. Then $M_{\downarrow X}$ is semisimple.

ii) Let N be a simple $F[X]$-module. Then $N^{\uparrow G}$ is semisimple.

Proof: i) Consider first the case $H = X$. Then i) is just a special case of Theorem 11.2 i). To prove ii) let V be a simple submodule of $N^{\uparrow G}$. Then $V_{\downarrow H} \simeq \bigoplus_i (N \otimes g_i)$ for suitable g_i's in G. As V is H-projective, it follows that for some i, $V | (N \otimes g_i)^{\uparrow G} \simeq N^{\uparrow G}$, as H is normal. Thus V is a direct summand of $N^{\uparrow G}$, and ii) follows by induction.

Now let $H \leq X \leq G$ be arbitrary. As N is H-projective, $N | (N_{\downarrow H})^{\uparrow X}$. Moreover, $N_{\downarrow H}$ is semisimple by what we just proved. Hence there exists a simple submodule W of $N_{\downarrow H}$ such that $N | W^{\uparrow X}$. Thus the general case ii) follows from the special which states that $W^{\uparrow G}$ is semisimple. Finally, the general case i) follows from i) and ii) in the special case and Mackey decomposition.

We now turn to the main object of this section, which is the decomposition of modules induced from a normal subgroup. The following short and elegant treatment goes back to Tucker (1965), Conlon (1964), Ward (1968) and Cline (1972). We have been inspired by Knörr's (1979) beautifully short and precise repetition of this, which again is inspired by Dade's conception of Clifford theory. For alternative sources, the reader may therefore want to consult Dade (1970) and in particular (1980).

We return to the notation of Definition 11.1.

Lemma 11.4. Assume $T = H$. Then $N^{\uparrow G}$ is indecomposable.

Proof: Assume $N^{\uparrow G} = M_1 \oplus M_2$. Then $M_{i\downarrow H} \simeq \bigoplus_{g_i \in \Gamma_i} (N \otimes g)$, $i = 1, 2$, for suitable subsets Γ_1, Γ_2 of $H \backslash G$ with $\Gamma_1 \cap \Gamma_2 = \phi$ and $\Gamma_1 \cup \Gamma_2 = H \backslash G$. Let $g_i \in \Gamma_i$, $i = 1, 2$, and set $g_3 = g_1^{-1} g_2$. Then $M_1 \simeq M_1 g_3$. But $M_1 g_{3\downarrow H} \simeq \bigoplus_{g \in \Gamma_1} (N \otimes gg_3)$, a contradiction as $g_1 g_3 \in \Gamma_2$.

Thus it seems reasonable moreover in the following to assume that N is G-stable. For any $g \in H \backslash G$, let $\phi_g : N \longrightarrow N$ be an Θ-isomorphism such that $x \longrightarrow \phi_g(x) \otimes g$ is an $\Theta[H]$-isomorphism of N into $N \otimes g$. Then $\phi_g(xh) = \phi_g(x)ghg^{-1}$ for all $h \in H$.

Recall that

$$\text{(3)} \qquad Fr_H^G : (N, N^{\uparrow G})^H \longrightarrow (N^{\uparrow G}, N^{\uparrow G}) =: E$$

is an Θ-isomorphism, and

$$\text{(4)} \qquad Fr_H^G(\phi)(\textstyle\sum x_i \otimes g_i) = \sum \phi(x_i) g_i \quad .$$

Also, $(N, N^{\uparrow G})^H = \bigoplus_{g_i \in H \backslash G} (N, N \otimes g_i)^H$. Set

$$\text{(5)} \qquad E_{g_i} = Fr_H^G((N, N \otimes g_i)^H) \quad .$$

Thus $E = \bigoplus_i E_{g_i}$. Assume $g_1 = 1$, without loss of generality. Notice that E_1 is not only an Θ-space, but a ring, as

$$(6) \qquad Fr_H^G(\phi_1 \circ \phi_2) = Fr_H^G(\phi_1) \circ Fr_H^G(\phi_2)$$

for $\phi_1, \phi_2 \in (N, N \otimes 1)^H$. We identify E_1 with the ring $(N, N)^H$. Alternatively, any $\phi \in (N, N)^H$ extends uniquely to $\phi \in E$ by

$$(7) \qquad \phi(x \otimes g) = \phi(x \otimes 1)g \quad ,$$

and if $\phi \in E$, then $\phi \in E_1$ if and only if $\phi(x \otimes 1) \in N \otimes 1$.

For $g_j \in H \backslash G$, let $\psi_{g_j} \in E_{g_j}$ be the map obtained by applying Fr_H^G to the isomorphism $x \longrightarrow \phi_{g_j}(x) \otimes g_j$ of N onto $N \otimes g_j$. Then

$$(8) \qquad \psi_{g_j}(\sum_i x_i \otimes g_i) = \sum_i \phi_{g_j}(x_i) \otimes g_j g_i)$$

and thus $E_{g_j} = E_1 \psi_{g_j} = \psi_{g_j} E_1$. Moreover, if g', g'' are arbitrary, we may always for $\phi_{g'g''}$ choose $\phi_{g'} \circ \phi_{g''}$, which shows that $E_{g'} E_{g''} = E_{g'g''}$. (Notice that E_{g_i} does not depend on the choice of g_i.) Thus

Proposition 11.5. With this notation,

$$(9) \qquad E = \bigoplus_{g_i \in H \backslash G} E_1 \psi_{g_i}$$

and

$$(10) \qquad E_1 \psi_{g'} E_1 \psi_{g''} = E_1 \psi_{g'g''}$$

for all $g', g'' \in H \backslash G$. Also, E_1 is ring isomorphic to $E_1 \psi_1$ and we identify these rings. In particular, E is a free E_1-module, as E_1 is local, of rank $|G : H|$.

Remark: In parts of the literature, E is called a strongly graded Clifford algebra. However this does not really add anything to its properties.

Remark: We want to stress that it is not usually true that $\psi_{g'}\psi_{g''} = \psi_{g'g''}$ for all $g', g'' \in H \backslash G$.

In the following, we let (F, R, S) be a p-modular system and we let Θ equal R or F. Otherwise, we continue to use the notation and assumptions above.

Lemma 11.6. $J(E_1)E \subseteq J(E)$.

Proof: As $E_1\psi_{g_i} = \psi_{g_i}E_1$ for all i.

We may now state the first main result:

Let $e_1, \ldots, e_r \in E$ be primitive idempotents such that $\{U_i\}$, $U_i = e_iE$ form a set of representatives of the isomorphism classes of p.i.m.'s of E. Thus

(11)
$$E_E \simeq \bigoplus_{i=1}^{r} U_i^{(d_i)}$$

for uniquely determined $d_i \in \mathbb{N}$, and $U_i \simeq U_j$ if and only if $i = j$.

Theorem 11.7. With the notation and assumptions above,

(12)
$$N^{\uparrow G} \simeq \bigoplus_{i=1}^{r} M_i^{(d_i)}$$

where $M_i = e_i(N^{\uparrow G})$ is indecomposable and $M_i \simeq M_j$ if and only if $i = j$. Moreover, $M_i = U_i(N \otimes 1)$ if we consider $N^{\uparrow G}$ as a left

E-module in the usual way, and

(13) $$\operatorname{rank}_{\Theta}(M_i) = \operatorname{rank}_{\Theta}(N)\operatorname{rank}_{E_1}(U_i) \quad .$$

(Note that U_i is a free E_1-module, as E is and E_1 is local.)

Proof: (12) is an immediate consequence of (11), by Fitting's Theorem. Also, $e_i(N^{\uparrow G}) = U_i(N^{\uparrow G})$, as $U_i = e_iE$. However, as $N^{\uparrow G} = E(N \otimes 1)$, we also have that $U_i(N^{\uparrow G}) = U_i(N \otimes 1)$. In particular, $\leq$ holds in (13). Summing over all indecomposable components of $N^{\uparrow G}$, we deduce that then equality in fact must hold.

For $\mathfrak{X}$ a family of subgroups of G, we set $E_{\mathfrak{X}} = (N^{\uparrow G}, N^{\uparrow G})^G_{\mathfrak{X}}$. We then have

Proposition 11.8 (Knörr (1979)). Let $\mathfrak{X}$ be any family of subgroups of G. Then

(14) $$E_{\mathfrak{X}} = (E_1)_{\mathfrak{X} \cap_G H} E \quad .$$

Proof: By Corollary 5.4 ii).

Assume now furthermore that $E_1/J(E_1) \simeq F$ which for instance is the case if N is absolutely indecomposable. Then

(15) $$E/J(E_1)E \simeq F^{\wedge}[G/H] \quad ,$$

a twisted group algebra of G/H over F. The twisting comes from

(16) $$\psi_{g_i}\psi_{g_j} \equiv \alpha(g_i, g_j)\psi_{g_ig_j} \bmod J(E_1)E$$

where the factor set $\alpha(g_i,g_j) \in F$ is not always 1.

In order to prove a very important theorem of Green (1959a), (1962b), we need the following well-known result:

Proposition 11.9. Let F be any perfect field of characteristic p, and let P be a p-group. Then any twisted group algebra $\hat{F}[P]$ is isomorphic to the ordinary group algebra $F[P]$.

Proof: Anybody familiar with cohomology will of course be aware of a proof of this fact. If F is perfect, it follows from the fact that $H^2(P, F^{\#}) = 0$. And if F is finite it is of course therefore as well a consequence of Schur's Theorem on the existence of compliments in finite groups to normal subgroups of order prime to their index. If F is not finite, the same argument of course works if the multiplication coefficients lie in a finite field. However, for the sake of completeness, we will give a self-contained proof (which of course is cohomology in disguise).

As remarked, we must assume that F is perfect. Now multiplication in $\hat{F}[P]$ is given by

(17) $$x_1 \cdot x_2 = \alpha(x_1, x_2) x_1 x_2$$

for $x_1, x_2 \in P$. For every $x \in P$, we set

(18) $$f(x) = \Big(\prod_{z \in P} \alpha(x, z)\Big)^{-\frac{1}{|P|}}$$

which is well defined as F is perfect. Now the associativity of $\hat{F}[P]$ implies that for all $x, y, z \in P$,

(19) $$\alpha(y, z)\alpha(x, yz) = \alpha(xy, z)\alpha(x, y) \quad .$$

Taking the $-\frac{1}{|P|}$'th root on both sides and then the product for all z, we get

$$(20) \qquad f(x)f(y) = f(xy)\alpha(x, y)^{-1} .$$

But then $\{f(x)x, \cdot\}$ is a group isomorphic to P, where the isomorphism is given by $x \longrightarrow f(x)x$, since

$$(21) \qquad f(x)x.f(y)y = f(xy)xy$$

by (17) and (20), and we are done.

As an immediate consequence of this and (15) we now obtain (recall that Θ equals F or R)

Theorem 11.10 (Green (1959a)). Let $H \trianglelefteq G$ with $G/H \simeq \mathbf{Z}_p$. Let N be an absolutely indecomposable $\Theta[H]$-module. Then $N^{\uparrow G}$ is absolutely indecomposable.

Proof: We may as well assume that F is algebraically closed. Now, by Lemma 11.4 we may also assume that N is G-stable. Using the notation above, we therefore have that

$$(22) \qquad E/J(E_1)E \simeq F[\mathbf{Z}_p] .$$

Hence $E/J(E) \simeq F[\mathbf{Z}_p]/J(F[\mathbf{Z}_p]) \simeq F$ and it follows that $N^{\uparrow G}$ is indecomposable. (For definition of absolutely indecomposable, see App. III.)

Remark: It is of vital importance to assume that N is absolutely indecomposable. For an example to illustrate this, take (F, R, S) such that S does not contain a third root of unity and $F \simeq GF(2)$, which is possible. Let G be the symmetric group of degree 3, and H the normal subgroup of order 3. For N, take an R-form of the 2-dimensional simple S[H]-module. Then $N^{\uparrow G} \simeq M \oplus M$, where $M_{\downarrow H} \simeq N$.

We mention a couple of very important consequences, all due to Green.

Corollary 11.11. Let $H \leq G$ and assume there exists a chain of subgroups

$$G = G_1 > G_2 > \dots > G_n = H \tag{23}$$

such that $G_{i+1} \triangleleft G_i$, for all $i = 1, \dots, n-1$ and $|G : H|$ is a power of p. Let N be an absolutely indecomposable $\Theta[H]$-module. Then $N^{\uparrow G}$ is absolutely indecomposable.

Proof: An immediate consequence of Theorem 11.10 and induction.

Corollary 11.12. Let G be a finite group and let M be an indecomposable $\Theta[G]$-module. Let V be a vertex of M, and let $V \leq Q \in \mathrm{Syl}_p(G)$. Then $|Q : V|$ divides $\mathrm{rank}_\Theta(M)$.

Proof: By Theorem 3.3 i), each direct summand of $M_{\downarrow Q}$ has a vertex of order at most $|V|$. Thus it suffices to consider the case $G = Q$. Also, by Lemma 3.10 it suffices to consider the case where $\Theta = F$. Let $\overline{F}$ be the algebraic closure of M, and let $N \in \mathbf{M}_F(V)$ be a source of M. Then $(N \otimes_F \overline{F})^{\uparrow Q} = N^{\uparrow Q} \otimes_F \overline{F}$, and thus $M \otimes_F \overline{F}$ is a direct summand of this module. However, any direct summand of $(N \otimes_F \overline{F})^{\uparrow Q}$ has a dimension divisible by $|Q : V|$ by Green's Theorem 11.10.

Corollary 11.13. Let $H \trianglelefteq G$ and assume G/H is a p-group. Assume moreover that F is a splitting field of $F[G]/J(F[G])$ and let N be a simple $F[H]$-module which is absolutely indecomposable such that its inertial group equals G. Then there exists a simple $F[G]$-module M, uniquely determined up to isomorphism, such that $M_{\downarrow H} \simeq N$.

Proof: As $(N^{\uparrow G})_{\downarrow H} \simeq N^{(|G:H|)}$ by assumption, any simple composition factor M of $N^{\uparrow G}$ has the property that $N | M_{\downarrow H}$. Let P_0 denote the projective $F[H]$-cover of N, P the projective $F[G]$-cover of M. By our assumption on N, P_0 is absolutely indecomposable. Hence $P_0^{\uparrow G}$ is an indecomposable projective $F[G]$-module by Theorem 11.10. By The Nakayama Relations, $M \subseteq P_0^{\uparrow G}$ and thus $P_0^{\uparrow G} = P_1$. In particular, any composition factor of $N^{\uparrow G}$ is isomorphic to M. Moreover,

$$(M, N)^H \simeq (M, N^{\uparrow G}) \simeq F \tag{24}$$

as $N^{\uparrow G} \subseteq P_1$, which proves that $M_{\downarrow H} \simeq N$, as $M_{\downarrow H} \simeq N^{(m)}$ for some m.

Remark: In other cases, where no information on G/H is available, it is more difficult to take advantage of the important consequence (15) of Theorem 11.7. However, from the theory of projective representations of finite groups (in the sense of Schur, see Curtis & Reiner (1982), §11), it does follow that d_i of (11) divides $|G/H|$ for all i, still under the assumption of course that $E_1/J(E_1) \simeq F$. Also we wish to point out that if $|G/H|$ is prime to p, then $J(E) = J(E_1)E$. In this case we moreover have that $M_{i\downarrow H} \simeq N^{(d_i)}$ by (13), as U_1 is simple then.

In the next chapter we shall demonstrate how to deduce a deep result on vertices of simple modules and R-forms of irreducible characters (see Knörr (1979)) from Theorem 11.7.

For other aspects of Theorem 11.10, see Broué (1976), where the assumption on the field is weakened.

Our last result in this section is a new result due to Alperin, Collins & Sibley (1983) which will provide us with a very conceptual proof of the striking fact that the determinant of the Cartan matrix is a power of p, where p is the characteristic of the field involved.

In the following we let $H \triangleleft G$, and let F be a field of characteristic p. Set $\overline{G} = G/H$ and let $E_1, \ldots, E_\ell$ represent the isomorphism classes of simple $F[\overline{G}]$-modules. Denote the projective $F[\overline{G}]$-cover of E_i by $\overline{P}_i$ and the projective $F[G]$-cover of E_i by P_i.

Set $J = J(F[H])$ and let A be the augmentation ideal of $F[H]$. As $H \triangleleft G$, $F[H]$, J and A are all $F[G]$-modules, where the action is given by conjugation.

Theorem 11.14. With the notation above,

$$\overline{P}_j \otimes J^i/AJ^i \simeq P_jJ^i/P_jJ^{i+1} \tag{25}$$

as $F[G]$-modules, for all i,j.

Proof: Our basic assumptions yield the existence of a surjective $F[G]$-homomorphism $\mu : P_j \longrightarrow \overline{P}_j$ where $\overline{P}_j$ is viewed as an $F[G]$-module by inflation. As H acts trivially on $\overline{P}_j$ it follows that $P_jA \subseteq \operatorname{Ker} \mu$. On the other hand, P_j/P_jA is obviously an $F[\overline{G}]$-module, and indecomposable as such, as P_j is an indecomposable $F[G]$-module. In particular, we have

Lemma 11.15 (Willems (1980)). With the notation above, $\bar{P}_j$ and P_j/P_jA are isomorphic as $F[\bar{G}]$-modules.

Next we consider the projective $F[H]$-cover Q of the trivial $F[H]$-module I_H. As $H \trianglelefteq G$, $Q \simeq Q \otimes g$ as $F[H]$-modules for all $g \in G$. Thus $(Q^{\uparrow G})_{\downarrow H} \simeq Q^{(|\bar{G}|)}$. Moreover, as $Q^{\uparrow G}$ maps onto $I_H^{\uparrow G} \simeq F[\bar{G}]$ which in turn maps onto E_j, P_j is isomorphic to a direct summand of $Q^{\uparrow G}$. In particular, $P_{j\downarrow H} \simeq Q^{(m_j)}$ for some $m_j \in \mathbb{N}$. By definition, $QJ = QA$ and thus $(P_jJ)_{\downarrow N} \simeq (P_jA)_{\downarrow N}$. As $P_A \supseteq PJ$, it follows that $P_jA = P_jJ$. Thus $\bar{P}_j \simeq P_j/P_JJ$ as $F[G]$-modules. Consider the map

$$(26) \qquad P_j/P_jJ \otimes J^i/AJ^i \longrightarrow P_jJ^i/P_jJ^{i+1}$$

defined by

$$(27) \qquad (x + P_jJ) \otimes (y + AJ^i) \longrightarrow xy + P_jJ^{i+1}$$

for $i \geq 1$. This is obviously well defined, F-linear and surjective. Moreover, for all $g \in G$,

$$(28) \qquad \begin{aligned} &((x + P_jJ) \otimes (y + AJ^i)g \\ &= (xg + P_jJ) \otimes (g^{-1}yg + AJ^i) \longrightarrow xyg + P_jJ^{i+1} \\ &\qquad\qquad = (xy) + P_jJ^{i+1})g \end{aligned}$$

and thus the map of (26) is an $F[G]$-homomorphism. It remains to prove that the map is injective. But this will follow from the surjectivity if the two modules have the same dimension. However if we in particular consider them as $F[H]$-modules and recall that $P_{j\downarrow H} \simeq Q^{(m_j)}$, it suffices to prove that

(29) $$\dim_F(Q/QJ \otimes J^i/AJ^i) = \dim_F(QJ^i/QJ^{i+1}) \quad .$$

But

(30) $$\dim_F(Q/QJ \otimes J^i/AJ^i) = \dim_F(J^i/AJ^i)$$
$$= \dim_F(J^i/J^{i+1}, I_H)$$

which equals the right-hand side of (29) by Lemma 1.5.

For a number of other relations between projective modules of F[G] and F[G/H] see Willems (1980) and Huppert & Willems (1975).

12. Permutation modules.

Let Θ be a principal ideal domain. If $H \leq G$, we let I_H denote the trivial $\Theta[H]$-module, 1_H the trivial character of H.

Definition 12.1. By a transitive permutation module over Θ we understand a module in $\mathbf{M}_\Theta(G)$ isomorphic to $I_H{}^{\uparrow G}$ for some $H \leq G$. Alternatively, it is a module isomorphic to a (right) principal ideal of $\Theta[G]$ generated by an element of the form $\sum_{h \in H} h$.

A permutation module is a module isomorphic to a direct sum of transitive permutation modules.

Assume Krull-Schmidt holds for $\Theta[G]$-modules. By a trivial source module (over Θ) we understand an indecomposable direct summand of a permutation module (over Θ).

The basic (and trivial) properties of permutation modules can be expressed most smoothly if we consider them as elements of the Green ring:

Lemma 12.2. Let $B(G)$ denote the vector space spanned by all permutation modules in the complex Green ring. Then $B(G)$ is a ring, called the Burnside ring. Moreover

i) $i_{H,G}(B(H)) \subseteq B(G)$.

ii) $r_{G,H}(B(G)) \subseteq B(H)$.

Proof: Closure under multiplication follows from the Mackey tensor product theorem.

The following important results show that trivial source modules are much easier to work with than modules at large.

Lemma 12.3. Let $H, K \leq G$. Then

(1) $$\operatorname{rank}_{\Theta}(I_H{}^{\uparrow G}, I_K{}^{\uparrow G})^G = (1_H{}^{\uparrow G}, 1_K{}^{\uparrow G})_G = |H\backslash G/K| .$$

Proof: This is just an application of the Nakayama relations:

(2) $$(I_H{}^{\uparrow G}, I_K{}^{\uparrow G})^G \simeq (I_H{}^{\uparrow G}{}_{\downarrow K}, I_K)^K$$
$$\simeq \sum_{g_i \in H\backslash G/K} ((I_{H^{g_i} \cap K})^{\uparrow K}, I_K)^K$$
$$\simeq \sum_{g_i \in H\backslash G/K} (I_{H^{g_i} \cap K}, I_{H^{g_i} \cap K})^{H^{g_i} \cap K}$$

which obviously is of dimension $|H\backslash G/K|$. The second equality follows by choosing Θ to be a field of characteristic 0.

As promised in Ch. I, Section 14, we now prove, as a corollary of Lemma 12.3,

Theorem 12.4. Let (F, R, S) be a p-modular system. Then

i) (Scott (1973)) The endomorphism ring of a transitive permutation module of $F[G]$ is liftable.

ii) A trivial source $F[G]$-module is liftable, to a trivial source $R[G]$-module.

iii) Let A and B be arbitrary trivial source modules of $R[G]$, and set $\bar{A}$ resp. $\bar{B}$ equal to A/A^{-} resp. $B/B\pi$. Then

$$(\bar{A}, \bar{B})^G = (A, B)^G/(A, B)^G\pi \quad . \tag{3}$$

In other words, $F[G]$-homomorphisms between trivial source modules (even if they originate from different permutation modules) are liftable.

Proof: Let $H, K \leq G$. From Lemma 13.2, we get that

$$(\bar{I}_H{}^{\uparrow G}, \bar{I}_K{}^{\uparrow G})^G = (I_H{}^{\uparrow G}, I_K{}^{\uparrow G})/(I_H{}^{\uparrow G}, I_K{}^{\uparrow G})\pi \quad . \tag{4}$$

In particular, if $H = K$, the idempotents of $(\bar{I}_H{}^{\uparrow G}, \bar{I}_H{}^{\uparrow G})^G$ and $(I_H{}^{\uparrow G}, I_H{}^{\uparrow G})$ are in one-to-one correspondence to each other by Theorem I.11.2, hence so are the direct summands of $\bar{I}_H{}^{\uparrow G}$ and $I_H{}^{\uparrow G}$, and i), ii) and iii) all follow.

Remark: In case $H = K$, we may also prove i) above by simply finding a basis, which goes back to Schur, of the endomorphism ring. This is Scott's approach, and we will return to that later.

In the following, we first discuss the basic properties of trivial source modules. We therefore continue with the notation of Theorem 12.4, and let $\mathcal{O}$ equal F or R.

Lemma 12.5. Let $M \in \mathbf{M}_{\mathcal{O}}(G)$ be indecomposable and let V be a vertex of M. Then M is a trivial source module if and only if

I_V is a source of M.

Proof: This is easy to check. But since it seems so powerful let us just briefly comment on two different ways of seeing this. The first is simply that otherwise we obviously would have chosen another name. The second is Mackey decomposition.

Example 1. Let V be a p-group, let N be an indecomposable projective $\Theta[N_G(V)/V]$-module and inflate it to an $\Theta[N_G(V)]$-module. Then any indecomposable direct summand of $N^{\uparrow G}$, in particular its Green correspondent, is a trivial source module.

For the following lemma, see Landrock (1981b) and Damgård (1983). Also, ii) below was first observed by Scott (1973), but the proof is more involved.

Lemma 12.6. Let $\hat{M} \in \mathbf{M}_R(G)$ be a trivial source module. Thus $M := \hat{M}/\hat{M}\pi$ is the corresponding trivial source module for $F[G]$. Let $\chi_{\hat{M}}$ be the character of $M \otimes_R S$.

i) Let $Q \leq G$ be a p-group. Then

$$\dim_F(\mathrm{Soc}(M_{\downarrow Q})) = (\chi_{\hat{M}}, 1_Q)_Q \quad . \tag{5}$$

ii) Let $x \in G$ be a p-element. Then $\chi_{\hat{M}}(x)$ equals the number of indecomposable direct summands of $M_{\downarrow \langle x \rangle}$ isomorphic to $I_{\langle x \rangle}$. In particular, $\chi_{\hat{M}}(x)$ is a non-negative integer. Moreover,

iii) $\chi_{\hat{M}}(x) \neq 0$ if and only if x belongs to some vertex of M.

Proof: Let $M | I_H^{\uparrow G}$. Then any direct summand of $\overline{M}_{\downarrow Q}$ is of the form $(I_{H^g \cap Q})^{\uparrow Q}$ for some $g \in G$ by Mackey decomposition and Example 2 of Section 6. Moreover, as we saw in that example,

$$(6) \qquad \dim_F(\mathrm{Soc}(I_{H^g \cap Q})^{\uparrow Q}) = 1 = ((1_{H^g \cap Q})^{\uparrow Q}, 1_Q)_Q$$

and $(1_{H^g \cap Q})^{\uparrow Q}$ is the character of $(\hat{I}_{H^g \cap Q}{}^{\uparrow Q}) \otimes_R S$. This proves i). Choosing $Q = \langle x \rangle$ we see moreover that $(1_{H^g \cap Q})^{\uparrow Q}(x) = 0$ unless $H^g \cap Q = Q$, in which case it equals 1, and ii) follows. Finally, choosing $H = V$ in view of Lemma 12.5, where V is a vertex of M, and $Q = \langle x \rangle$ we get $H^g \cap Q = Q$ for some $g \in G$ if and only if $x \in H^g$ and iii) is proved.

Lemma 12.7. Let $H \leq Q$. Then

i) $I_H{}^{\uparrow G}$ has exactly one factor module and exactly one submodule isomorphic to I_G.

ii) The indecomposable direct summand $P(H)$ of $I_H{}^{\uparrow G}$ with I_G as a submodule is identical to that with I_G as a factor module.

iii) Let $H_i \leq G$, $i = 1, 2$. Then $P(H_1) \simeq P(H_2)$ if and only a Sylow p-subgroup of H_1 is G-conjugate to one in H_2.

iv) The Sylow p-subgroups of H are vertices of $P(H)$.

Proof: i) follows directly from the Nakayama relations.

ii) Let χ_H denote the character of $I_H{}^{\uparrow G}$. Then $(\chi_H, 1_G)_G = 1$. As every direct summand of $I_H{}^{\uparrow G}$ is liftable, there is precisely one indecomposable direct summand $P(H)$ of $I_H{}^{\uparrow G}$, whose character has the trivial character of G as a constituent. But then $P(H)$ has the claimed property by Theorem I.17.3.

iii) is an immediate consequence of ii) and Lemma 12.5, as $I_H | I_Q{}^{\uparrow H}$ where $Q \in \mathrm{Syl}_p(H)$.

iv) follows from iii).

We now concentrate on the endomorphism ring of a transitive permutation module.

For notation, let $H \leq G$ and set $\Delta = \{Hx_i\}$, where $x_i \in H \backslash G$ and $x_1 \in H$. Let $\{\Delta_j\}$ denote the orbits of G on $\Delta \times \Delta$. For $X \in \Delta$, set

(8) $$\Delta_j(X) = \{Y \in \Delta \mid (X, Y) \in \Delta_j\} \quad .$$

Recall there is a one-to-one correspondence between double cosets of H in G and the orbits of G on $\Delta \times \Delta$. Indeed, if $H \backslash G/H = \{\gamma_j\}$, and $X_i = Hx_i$, then $(X_1, X_1\gamma_j)$ is a full set of representatives of orbits on $\Delta \times \Delta$. We therefore choose notation so that $(X_1, X_1\gamma_j) \in \Delta_j$. It now easily follows that if $X = X_1 g$, then

(9) $$\Delta_j(X) = \{X_1\gamma_j h \mid h \in (H \cap H^{\gamma_j}) \backslash H\}g \quad .$$

In particular, $k_j := |\Delta_j(X)| = |H : H \cap H^{\gamma_i}|$.

We now define $A_j \in (\mathbf{Z}\Delta, \mathbf{Z}\Delta)$ where $\mathbf{Z}\Delta$ is the free $\mathbf{Z}$-span over Δ by

(10) $$A_j(X) = \sum_{Y \in \Delta_j(X)} Y$$

and extend linearly. It immediately follows that $A_j \in (\mathbf{Z}\Delta, \mathbf{Z}\Delta)^G$, as $\Delta_j(Xg) = \Delta_j(X)g$. We now have

<u>Lemma 12.8</u> (Schur (1933)). The A_j's form a basis of $(\mathbf{Z}\Delta, \mathbf{Z}\Delta)^G$.

<u>Proof</u>: Let $A \in (\mathbf{Z}\Delta, \mathbf{Z}\Delta)^G$. Then A is completely determined by $A(X_1)$. However, as $X_1 h = X_1$ for all $h \in H$, it follows that for arbitrary j, all $Y \in \Delta_j$ occur with the same multiplicity, and

we are done, as the A_j's obviously are linearly independent.

Next we recall that if $\sigma = \sum_{h \in H} h$, then

$$\mathbf{Z}\Delta \simeq (\mathbf{Z}\sigma)\uparrow_H^G \simeq \mathrm{span}_{\mathbf{Z}}\{\sigma x_i\} \simeq \sigma\mathbf{Z}[G] \tag{11}$$

as $\mathbf{Z}[G]$-modules, and the isomorphisms are given by $X_i \longrightarrow \sigma \otimes x_i \longrightarrow \sigma x_i$. This induces an action of A_j on $\sigma\mathbf{Z}[G]$, namely

$$A_j(\sigma) = \sigma\gamma_j\Big(\sum_{h \in (H \cap H^{\gamma_j})\setminus H} h\Big) = \sum_{x \in H\gamma_j H} x \ . \tag{12}$$

Thus

<u>Lemma 12.9</u>. $A_j(\sigma) = a_j\sigma$, where

$$a_j = \Big(\sum_{k \in H/(H \cap H^{\gamma_j})} k_{\gamma_j^{-1}}\Big)^{\gamma_i} \ . \tag{13}$$

<u>Proof</u>: As $\sigma = \sum_{h \in H} h = \Big(\sum_{k \in H/(H \cap H^{\gamma_j})} k_{\gamma_j^{-1}}\Big)\Big(\sum_{h \in H \cap H^{\gamma_j}} h_{\gamma_j^{-1}}\Big)$.

<u>Corollary 12.10</u>. Set $\mathfrak{A}(H) = \{a \in \mathbf{Z}[G] \mid a\sigma = 0\}$, $\mathfrak{L}(H) = \{a \in \mathbf{Z}[G] \mid a\sigma \in \sigma\mathbf{Z}[G]\}$. Then $\mathfrak{L}(H)$ is a subalgebra of $\mathbf{Z}[G]$, $\mathfrak{A}(H)$ is an ideal in $\mathfrak{L}(H)$, and

$$\mathfrak{L}(H)/\mathfrak{A}(H) \simeq (\mathbf{Z}\Delta, \mathbf{Z}\Delta)^G \ . \tag{14}$$

Moreover

$$\mathfrak{L}(H) = \mathfrak{A}(H) \oplus \mathrm{span}_{\mathbf{Z}}\{a_j\} \tag{15}$$

as a $\mathbf{Z}$-module.

In the following, we set $E = (\mathbf{Z}\Delta, \mathbf{Z}\Delta)^G$. Thus $E_S := E \otimes_{\mathbf{Z}} S = (S\Delta, S\Delta)^G$ while $E_F := E \otimes_{\mathbf{Z}} F = (F\Delta, F\Delta)^G$ because of the fact that Δ is G-invariant.

As $S\Delta$ is semisimple, E_S is a semisimple ring. If we moreover set $E_R = E \otimes_{\mathbf{Z}} S$, the relations between E_R, E_S and E_F are described in Ch. I, Sections 12, 14, and 15. In particular, we may speak of the blocks and the decomposition numbers of this triple, the study of which clearly is necessary in order to obtain more information about $F\Delta$.

Here we proceed to sketch how to decide the number of blocks of E_R. In the following chapter, we will investigate these blocks further. This was first done by Scott (1973).

The following is inspired by some lectures D. Benson gave on the subject, and Damgård (1983). We will assume that S is a splitting field of E_S or equivalently, that any direct summand of $S\Delta$ is absolutely indecomposable. Thus $(E_S)_{E_S} \simeq \bigoplus_{i=1}^{r} V_i^{\dim_S V_i}$, where V_i is a set of representatives of the simple E_S-modules. Similarly, as an S[G]-module,

$$(16) \qquad S\Delta \simeq \oplus X_i^{\dim_S V_i},$$

where X_i is a simple S[G]-module, and $X_i \simeq X_j$ if and only if $i = j$. If e_i is the central primitive idempotent of E_S corresponding to V_i, then $e(S\Delta) \simeq X_i^{\dim_S V_i}$. Moreover

$$(17) \qquad S\Delta \simeq \oplus V_i^{(\dim_S X_i)}$$

as a left E_S-module.

We will now relate this to our natural basis $\{A_j\}$ of E_R. First of all we define a pairing on the orbits of G on $\Delta \times \Delta$ simply by the following: If $(X, Y) \in \Delta_r$, we let $\Delta_{r'}$ be the orbit which contains (Y, X).

Now, let $Tr : E_S \longrightarrow S$ denote the trace-function on elements in E_S acting on $S\Delta$. Our first observation is that

$$\text{(18)} \qquad Tr(A_{r'} A_s) = |G : H| k_s \delta_{rs}$$

where we recall that $k_s = |\Delta_s(X)|$. This easily follows from our definition of A_s. Also, we let $Tr_i : E_S \longrightarrow S$ denote the trace function on elements of E_R acting on V_i. Obviously

$$\text{(19)} \qquad Tr \sim \sum_i \dim_S(V_i) Tr_i \quad .$$

<u>Proposition 12.11</u>. With the notation above,

i) $e_i = \dfrac{\dim_S(X_i)}{|G : H|} \left(\sum_r \dfrac{Tr_i(A_{r'})}{k_r} A_r\right)$

ii) $\dim_S(X_i) = \dfrac{|G : H| \dim_S(V_i)}{\sum_r Tr_i(A_r) Tr_i(A_{r'}) k_r^{-1}}$

iii) $\sum_r Tr_i(A_{r'}) Tr_j(A_r) k_r^{-1} = \delta_{ij} \dfrac{|G : H| \dim_S V_i}{\dim_S X_i}$.

<u>Proof</u>: Let $e = \sum a_r A_r$. By (17), (18) and (19),

$$\text{(20)} \qquad Tr(A_{r'} e_i) = |G : H| k_r a_r = \dim_S(X_i) Tr_i(A_{r'})$$

from which a_r is determined, and i) follows. In particular, as $Tr_i(e_i) = \dim_S(V_i)$, we get

(21) $$\dim_S(V_i) = \sum_r a_r \mathrm{Tr}_i(A_r) = \sum_r \frac{\dim_S(X_i)\mathrm{Tr}_i(A_{r'})\mathrm{Tr}_i(A_r)}{|G:H|k_r}$$

from which ii) follows. As evidently $\mathrm{Tr}_i(e_j) = \delta_{ij}\dim_S(V_i)$, iii) follows as well.

Example 1. Choose notation so that X_1 is the trivial 1-dimensional $S[G]$-module. It is easily computed that $\mathrm{Tr}_1(A_r) = k_r$ and

(22) $$e_1 = \frac{1}{|G:H|}\left(\sum_r A_r\right) .$$

Thus $e_1 \in E_k$ if and only if $|G:H|$ is prime to p, which is hardly any surprise.

Another interesting fact is

Proposition 12.12. With the notation above,

(23) $$\mathrm{Tr}_i(A_r) = \overline{\mathrm{Tr}_i}(A_{r'}) \quad ,$$

the complex conjugate.

Proof: (Damgård (1983). Introduce an inner product on $S\Delta$ such that Δ form an orthonormal basis. Hence each orthogonal complement of an $S[G]$-submodule of $S\Delta$ is an $S[G]$-module as well. In particular, $S\Delta$ decomposes into a direct sum of orthogonal simple $S[G]$-modules. Thus we may choose another orthonormal basis $\Gamma = \bigcup \Gamma_i$, where Γ_i is an orthonormal basis of $e_i(S\Delta)$. Let $\underline{A}_q$ denote the matrix of A_q w.r.t. Δ. Then there exists a unitary matrix U such that $U^{-1}\underline{A}_q U$ is the matrix of A_q w.r.t. Γ. However, as obviously $\underline{A}_{q'} = (\underline{A}_q)^{tr}$, it follows that

$$(24) \qquad \overline{(U^{-1}\underline{A}_q U)}^{tr} = \overline{U}^{tr}(\overline{\underline{A}}_q)^{tr}\overline{U}^{-1tr} = U^{-1}A_{q'}U$$

as $\overline{A}_q = A_{q'}$, and (26) follows.

<u>Example 2</u>: If $H = 1$, (23) simply states the well-known result that $\chi\overline{(g)} = \chi(g^{-1})$ for any irreducible character χ of G and any $g \in G$.

Our first observation is that if somehow we can find the character table $\{Tr_i(A_j)\}$, we at least know something about the possible components of $F\Delta$ other than just the fact that they are liftable. More precisely, just as in the case of a group algebra, the character table $\{Tr_i(A_j)\}$ of E_S determines the p-blocks of E_R completely. Indeed, we just imitate the proof of Theorem I.12.6.

For each i, we define the map $\omega_i : Z(E_S) \longrightarrow S$ by

$$(28) \qquad \omega_i(\sum_j S_j A_j) := \sum_j S_i(\dim_S V_i)^{-1} Tr_i(A_j) \quad .$$

Then ω_i is the central homomorphism associated with e_i, i.e. $\omega_i(\sum t_j e_j) = t_i$. Indeed, elements of $Z(E_S)$ act on V_i as scalar matrices, and the value of ω_i on such an element is obviously its eigenvalue on V_i. Moreover, as $A_j \in E_R$ and $Z(E_R) \otimes_R S = Z(E_S)$, ω_i induces an algebra homomorphism $\overline{\omega}_i : \overline{Z} \longrightarrow F$, where

$$(29) \qquad \overline{Z} = (Z(E_R) + E_R\pi)/E_R\pi \simeq Z(E_R)/Z(E_R)\pi$$

by Proposition 1.12.2. Finally, $\overline{Z}$ is an artinian algebra and $\overline{\omega}_i(J(\overline{Z})) = 0$ obviously, so $\overline{\omega}_i$ induces an algebra homomorphism $\overline{\overline{\omega}}_i$: $\overline{Z}/J(\overline{Z})) \simeq Z(E_F/J(E_F)) \longrightarrow F$ with the following property: Let $\varepsilon_1, \ldots, \varepsilon_m$ be the block idempotents of E_R and denote the corresponding elements of $\bar{Z}/J(\bar{Z})$ by $\{\bar{\varepsilon}_q\}$. Then $\bar{\omega}_i(\bar{\varepsilon}_q) = 1$ if $\varepsilon_q e_i = e_i$, 0 otherwise. In particular,

Proposition 12.13. The central primitive idempotents e_i and e_j of E_S lie in the same block if and only if ω_i and ω_j are equal to each other on $Z(E_R)$ modulo (π).

For results relating properties of permutation modules to the cohomology of G and its subgroups, see Scott (1976).

13. Examples

We are now ready to determine the Loewy structure of the p.i.m.'s of SL(2, 4) in characteristic 2, as promised in Example 1 of Section I.18. This goes back to Alperin (1972), the first paper to deal with this type of questions. For a detailed discussion of the structure of the p.i.m.'s of all simple groups with a dihedral Sylow 2-subgroup, see Erdmann (1977a).

Example 1. Let $G = SL(2, 4)$ and let F be a field of characteristic 2 containing GF(4). Let $U \in Syl_2(G)$ and let T be a complement in $H = N_G(U)$ to U. Then $H \simeq A_4$ and F[H] has 3 isomorphism classes of simple modules, all of dimension 1, which we will denote by I_1, 1 and 1^*. Note that they are the simple F[T]-modules. It immediately follows from Corollary 1.9.6 and Example I.10.3 that the corresponding p.i.m.'s of F[H] have Loewy series

$$\begin{array}{ccc} I_1 & 1 & 1^* \\ 1\ 1^* & I_1 1^* & I_1 1 \\ I_1 & 1 & 1 \end{array} \qquad (1)$$

Recall from Example I.18.1 that $F[G]$ has 4 isomorphism classes of simple modules, I, 2_1, 2_2 and 4.

For convenience we will denote the Brauer character of a module M by $\overline{M}$. The character tables now show that

(2) $$\overline{2}_{1\downarrow H} = \overline{2}_{2\downarrow H} = \overline{1}+\overline{1}^{*}, \quad 4_{\downarrow A} = 2\cdot\overline{I}_1+\overline{1}+\overline{1}^{*} .$$

As 2_1 and 2_2 are algebraically conjugate and their restriction to H cannot be semi-simple, it follows that the Loewy series of the restrictions are

(3) $$2_{1\downarrow H} = \begin{matrix} 1 \\ 1^{*} \end{matrix}, \quad 2_{2\downarrow H} = \begin{matrix} 1^{*} \\ 1 \end{matrix}, \quad 4_{\downarrow H} = P_{I_1}$$

without loss of generality where the last assertion follows from the fact that 4 is projective.

A vertex of I is U while 1 is a vertex of 4. As for 2_1 and 2_2, the dimension only tells us that the vertex is of order 2 or 4. Now we might cheat and say that 2_1 and 2_2 restricted to U are just the representations

(4) $$\left\{\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \middle| \rho \in GF(4)\right\}$$

and its algebraic conjugate, which shows that U is a vertex as otherwise the modules would be induced from a subgroup of order 2 by Green's Theorem 11.10. Hence Green Correspondence asserts that U is a vertex of 2_1 and 2_2.

We might also describe the indecomposable $F[H]$-modules with $\mathbf{Z}_2$ as vertex. We claim there is exactly one such module. Indeed, if $\tau \in U$ is an involution, then $I_{\langle\tau\rangle}^{\uparrow U}$ is indecomposable, and its inertial group in H is U. Thus $I_{\langle\tau\rangle}^{\uparrow H}$ is indecomposable. Using the Nakayama Relations we find that its Loewy series is

(5)
$$\begin{matrix} I_1 & 1 & 1^* \\ I_1 & 1 & 1^* \end{matrix}$$

Next we compute that

(6)
$$\bar{1}^{\uparrow G} = \bar{1}^{*\uparrow G} = \bar{I} + \bar{2}_1 + \bar{2}_2 \quad .$$

By Nakayama, $(1^{\uparrow G}, 2_2)^G = (2_1, 1^{*\uparrow G}) = F$, and $(1^*)^{\uparrow G} = (1^{\uparrow G})^*$. Thus $1^{\uparrow G}$ and $(1^*)^{\uparrow G}$ are indecomposable with Loewy series

(7)
$$\begin{matrix} 2_2 & & 2_1 \\ I & & 1 \\ 2_1 & & 2_2 \end{matrix}$$

We may now describe $\left(\begin{smallmatrix} 1 \\ * \\ 1 \end{smallmatrix}\right)^{\uparrow G}$. As U is T.I, Green Correspondence yields that this module is $2_1 \oplus P$, where P is projective. Now (7) yields that $P = P_{2_2}$ and it follows that P_{2_2}, and P_{2_1} have Loewy series as claimed in Example I.18.1. As $\mathrm{Ext}^1_{F[G]}(I, I) = 0$ by Corollary I.10.13, the structure of P_I is then completely determined by Corollary I.9.11.

We leave it to the reader to check that the Green Correspondent of the F[H]-module with $\mathbf{Z}_2$ as vertex has Loewy series

(8)
$$\begin{matrix} I \\ 2_1 2_2 \\ I \end{matrix}$$

which consequently is a trivial source module.

For a number of examples of the Loewy structure of the p.i.m.'s of some simple groups, see Benson (1983a & b), Erdmann (1977a), Landrock & Michler (1978 & 1980) and Schneider (1983b).

Example 2 (Illustration of Corollary 2.9). Recall from Example I.18.1 that SL(2, 4) has an irreducible character χ with the property that if (F, R, S) is a 2-modular system and $GF(4) \subseteq F$, then χ has R-forms A and B which reduced modulo 2 have Loewy series as in (7) above.

We see that $(\bar{A}, \bar{A})^G \simeq F$ is generated by the identity, and thus $(\bar{A}, \bar{A})_1^G = 0$ as predicted in Corollary 2.9. Also, $(\bar{A}, \bar{B})_1^G \simeq F$ generated by the map which sends the head 2_2 of $\bar{A}$ to the socle of $\bar{B}$. However, this map obviously factors through the injective hull P_{2_1} of $\bar{A}$. Thus $(\bar{A}, \bar{B})^G = (\bar{A}, \bar{B})_1^G$. Thus Corollary 2.9 cannot be improved.

Example 3 (Benson, Damgård (1983)). Let $G \simeq \Sigma_8$, acting in the usual way on $\{1, \ldots, 8\}$. Let Δ be the set of unordered triples of $\{1, \ldots, 8\}$. Let H be the stabilizer in G of the triple $A = \{6, 7, 8\}$. Thus $H \simeq \Sigma_5 \times \Sigma_3$.

Let us describe the corresponding permutation modules in characteristic 0 and 2. As we saw in Section 12, the orbits Δ_1 of G on $\Delta \times \Delta$ corresponds to the H-orbits on Δ. It is easy to see that H has 4 orbits on Δ consisting of all triples intersecting X in 3, 2, 1 or 0 points, $\Delta_1(X), \ldots, \Delta_4(X)$. We compute $k_1 = 1$, $k_2 = 15$, $k_3 = 30$ and $k_4 = 10$.

Next we describe the basis $\{A_1, \ldots, A_4\}$ of E_R: A_1 is the identity and for $Y \in \Delta$, $A_2(Y)$, $A_3(Y)$ and $A_4(Y)$ is the sum of all triples intersecting Y in 2, 1 and 0 points. Next a minor calculation will show that in the left regular representation of E_R, the matrices of A_2, A_3 and A_4 are

(9) $$\begin{Bmatrix} 0 & 15 & 0 & 0 \\ 1 & 6 & 8 & 0 \\ 0 & 4 & 8 & 3 \\ 0 & 0 & 9 & 6 \end{Bmatrix}, \begin{Bmatrix} 0 & 0 & 30 & 0 \\ 0 & 8 & 16 & 6 \\ 1 & 8 & 15 & 6 \\ 0 & 9 & 18 & 3 \end{Bmatrix}, \begin{Bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 6 & 4 \\ 0 & 3 & 6 & 1 \\ 1 & 6 & 3 & 0 \end{Bmatrix}$$

(Note that the sum of each row of A_i is k_i). Next we observe that these matrices commute. Thus E_R is commutative and the irreducible representations $\omega_1, \ldots, \omega_4$ of E_S are 1-dimensional. Let V_i be the module corresponding to ω_i. Then $\omega_j(A_i) = \mathrm{Tr}_j(A_i)$ by Sec. 12, (20), and the corresponding idempotents $e_1, \ldots, e_4$ are simply the eigenvectors with eigenvalues $\mathrm{Tr}_j(A_i)$ of A_i. We compute

(10)

$\mathrm{Tr}_j(A_i)$	e_1	e_2	e_3	e_4
ω_1	1	15	30	10
ω_2	1	7	-2	-6
ω_3	1	1	-5	3
ω_4	1	-3	3	-1

Notice that if E_S is not abelian it can be quite a task to find $\mathrm{Tr}_j(A_i)$!

By Proposition 12.13, (10) above shows that E_R has precisely two blocks and the block idempotents are $e_1 + e_2$ and $e_3 + e_4$.

Proposition 12.11 allows us to compute the dimensions of the irreducible constituents of $1_H^{\uparrow G}$, where 1 is the trivial character. They are

(11) $$1, \quad 7, \quad 20, \quad 28 \quad .$$

Thus if I_H is the trivial $F[H]$-module, $I_H^{\uparrow G}$ is the direct sum of 2 indecomposable modules of dimension 8 and 48, and each module has an endomorphism ring of dimension 2.

Finally we note that Proposition 12.11 yields an expression of the e_j's as linear combinations of the A_i's.

CHAPTER III. BLOCK THEORY

1. Blocks, defect groups and the Brauer map.

The approach to the theory of blocks in this and the following section is inspired from Alperin and Broué (1979). Only, we make the setup a little more general to prove some of Brauer's Main Theorems and the main results of Scott (1973) on endomorphism rings of permutation modules at the same time (see Damgård (1983)). The reader will hopefully find that the similar nature of the approaches in the present and the previous chapter makes the results very compatible. For an even more general setup, we refer to Puig (1981), where a number of the main results in the present as well as the previous chapter are proved at the same time.

Of course the majority of the results in many of the sections of the present chapter really goes back to Brauer. But it has been essential for us to present this from a quite different angle, which will allow us to advance further in certain directions.

In the following, G will be a finite group and θ a principal ideal domain at least for a start. Moreover, A will be an θ-algebra with a basis Δ, which is closed under multiplication, on which G acts such that $(xy)g = (x)g(y)g$ for all $x, y \in \Delta$. The situations we have in mind are

i) Δ is a normal subgroup of G (including G itself as the most important case) and G acts by conjugation

ii) $A = (\theta\Omega, \theta\Omega)$, where G acts on the set Ω and $\theta\Omega$ is the free θ-module with the elements of Ω as a basis. As the basis Δ for A we may pick $\Delta = \{\delta_{xy}\}_{x,y \in \Omega}$, where $\delta_{xy}(x) = y$ while $\delta_{xy}(z) = 0$ for $z \neq x$, which admits a natural action of G.

Recall from Chapter II, Section 2, that

(1) $$A^G := \{a \in A \mid (a)g = a\}.$$

The set of orbits of G on Δ is denoted by $Cl(G|\Delta)$. For $\Delta_i \in Cl(G|\Delta)$, set $[\Delta_i] = \sum_{x \in \Delta_i} x$. Thus

(2) $$A^G = \mathrm{Span}_\theta\{[\Delta_i] \mid \Delta_i \in Cl(G|\Delta)\}.$$

We now have all the results of Chapter II, Section 1, at our hand. To refresh our sweet memories, we restate Theorems II.1.7, II.1.8 and Corollary II.1.10. Recall that if $H \leq G$ and $a \in A^H$, then $Tr_H^G(a) = \sum_{g \in H\backslash G} (a)g$ and $Tr_H^G(A^H)$ is denoted by A_H^G.

<u>Proposition 1.1</u>. Let $H, K \leq G$. Then

i) Let $a \in A^K$. Then

(3) $$Tr_K^G(a) = \sum_{g_i \in K\backslash G/H} Tr^H_{g_i^{-1}Kg_i \cap H}((a)g_i)$$

ii) Let $a \in A^K$, $b \in A^H$. Then

(4) $$Tr_K^G(a)\, Tr_H^G(b) = \sum_{g_i \in K\backslash G/H} Tr^G_{g_i^{-1}Kg_i \cap H}((a)g_i b)$$

iii) A_H^G is an ideal in A^G.

<u>Definition 1.2</u>. Let $\Delta_i \in Cl(G|\Delta)$. By a <u>defect</u> group $D(\Delta_i)$ of Δ_i on G, we mean a Sylow p-subgroup of the stabilizer $C_G(x)$ in G of some $x \in \Delta_i$. Thus the defect groups of Δ_i in G form a G-conjugacy class.

<u>Lemma 1.3</u>. Let $\theta = F$ be a field of characteristic p and let $H \leq G$. Then

(5) $$\{[\Delta_i] \mid \Delta_i \in Cl(G|\Delta), D(\Delta_i) \underset{G}{\leq} H\}$$

is a basis for A_H^G.

<u>Proof</u>: Let $\Delta_i \in Cl(G|\Delta)$, and let $\Delta_i = \bigcup_{j=1}^{r_i} \Delta_{ij}$, disjoint

union, where $\Delta_{ij} \in Cl(H|\Delta)$. Let $x_{ij} \in \Delta_{ij}$. Then

(6) $$A^H = \mathrm{Span}_F\{\mathrm{Tr}^H_{C_G(x_{ij}) \cap H}(x_{ij})\}_{i,j}$$

and thus

(7) $$A^G_H = \mathrm{Span}_F\{\mathrm{Tr}^G_H(\mathrm{Tr}^H_{C_G(x_{ij}) \cap H}(x_{ij}))\}_{i,j}$$
$$= \mathrm{Span}_F\{\frac{|C_G(x_{ij})|}{|C_G(x_{ij}) \cap H|}[\Delta_i]\}_i$$
$$= \mathrm{Span}_F\{[\Delta_i] \mid D(\Delta_i) \underset{G}{\leq} H\}$$

as char $F = p$.

We now turn to the primitive idempotents of A^G. We continue to let $\mathcal{O} = F$ be a field of characteristic p.

Lemma 1.4. Let $e \in A^G$ be a primitive idempotent. Then there exists a p-subgroup D of G such that for any subgroup K of G, $e \in A^G_K$ if and only if $D \underset{G}{\leq} K$.

Proof: By Lemma 1.3, $A^G = A^G_P$ for $P \in Syl_p(G)$. Now choose $D \leq G$ with $|D|$ minimal such that $e \in A^G_D$. Assume $e \in A^G_K$ for some $K \leq G$. Then

(8) $$e = e^2 \in \sum_{g_i \in D\backslash G/K} A^G_{g_i^{-1}Dg_i \cap K}$$

by Proposition 1.1 ii). Hence $e \in A^G_{g_i^{-1}Dg_i \cap K}$ for some i by Rosenberg's Lemma II.3.9. By choice of D, $g_i^{-1}Dg_i \leq K$ then. Choosing $K = P$ we furthermore see that D must be a p-group.

Definition 1.5. Same notation as in Lemma 1.4. Any such group D is called a defect group of e or eA in G. Thus the defect groups of e in G form a G-conjugacy class. If $|D| = p^d$, then d is called the defect of e in G.

Corollary 1.6. Same notation as in Lemma 1.4. Let D be a defect group of e. Then $Z(eA) = eA^G_D$.

Proof: As $Z(eA) = eZ(A)$ and $e \in A_D^G$, we have that $Z(eA) \subseteq eA_D^G$. The other inclusion is obvious.

Lemma 1.7. Let $e \in A^G$ be a primitive idempotent with D as defect group in G. Then $e = \sum_i \alpha_i[\Delta_i]$, where the sum is over those $\Delta_i \in Cl(G|\Delta_i)$ with $D(C_i) \leq_G D$. Furthermore, there exists a Δ_0 with $\alpha_0 \neq 0$ and $D(\Delta_0) =_G D$.

Proof: The first part follows from Lemma 1.3, which also shows that $e \in \sum_{\alpha_i \neq 0} A^G_{D(\Delta_i)}$. Thus $e \in A^G_{D(\Delta_0)}$ for some Δ_0 by Rosenberg's Lemma I.3.9, which shows the last part.

Our next tool is the so-called Brauer map. Again the idea is Brauer's but we shall need a slightly more general setup (See Broué (1979). For an even more general setup, see Green (1978a) or Puig (1981).) First we observe

Lemma 1.8. Let $P \leq G$ be a p-group. Then

$$A^P = F\Delta^P \oplus \sum_{R<P} A_R^P \tag{9}$$

where Δ^P is the set of fixpoints of P in Δ.

Proof: As $F\Delta^P$ is spanned by the fixpoints of P on Δ and $\sum_{R<P} A_R^P$ by the other orbits, the sum certainly is direct by Lemma 1.3.

Definition 1.9. Let θ an arbitrary principal ideal domain. The θ-linear map $Br_P : A \to \theta\Delta^P$ is defined by

$$Br_P(x) = \begin{cases} x & \text{if } x \in \Delta^P \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

and extended linearly.

Again we return to the case where $\theta = F$ is a field of characteristic p.

Lemma 1.10. Let $P \leq H \leq G$, P a p-group. Then

(11) $$\mathrm{Ker}(Br_P) \cap A^H = \sum_{\substack{Q \leq H \\ P \not\leq_H Q}} A_Q^H$$

Proof: It suffices to consider the case $H = G$, obviously. Now, $A^G = A_1 \oplus A_2$, where

(12)
$$A_1 = \mathrm{Span}_F\{[\Delta_i] \mid \Delta_i \in \mathrm{Cl}(G|\Delta),\ P \leq_G D(\Delta_i)\}$$
$$A_2 = \mathrm{Span}_F\{[\Delta_i] \mid \Delta_i \in \mathrm{Cl}(G|\Delta),\ P \not\leq_G D(\Delta_i)\} .$$

Thus $A_2 \subseteq \mathrm{Ker}(Br_P)$, while the restriction of Br_P to A_1 is injective. Hence $A_2 = \mathrm{Ker}(B_P)$ and we are done by Lemma 1.3.

In particular,

Corollary 1.11. The restriction of Br_P to A^P is just the projection onto $F\Delta^P$ with kernel $\sum_{R<P} A_R^P$, which is an algebra homomorphism.

Proof: The first part is an immediate consequence of Lemma 1.10, while the last part follows from the fact that $\sum_{R<P} A_R^P$ is an ideal.

Corollary 1.12. Let $e \in A^G$ be a primitive idempotent and D a defect group of e in G. Let $P \leq G$ by any p-group. Then $Br_P(e) \neq 0$ if and only if $P \leq_G D$.

Proof: We must show that $e \in \mathrm{Ker}(Br_P)$ if and only if $P \not\leq_G D$. If $Br_P(e) = 0$ it follows from Lemma 1.10 and Rosenberg's Lemma II.3.9 that $e \in A_Q^G$ for some $Q \leq G$ with $P \not\leq_G Q$. But then $D \leq_G Q$ by Lemma 1.4 and thus $P \not\leq_G D$. The converse follows from Lemma 1.6 and the definition of Br_P.

Let us next justify why we have worked solely in characteristic p. Let (F,R,S) be a p-modular system, let A be an R-algebra with basis Δ, with G acting on Δ, and set $\overline{A} = A/A\pi$. It is then no longer possible to prove Lemma 1.3 as stated. Indeed the proofs show that we have to allow linear combinations of other orbit sums as well. However, whenever the sum of an orbit, whose defect groups are not G-conjugate to a subgroup of H, occurs, the coefficient will lie in (π). This corresponds to the fact we observed in Lemma II.2.10. Likewise, the Brauer map restricted to A^P is no longer an algebra homomorphism, simply because the decomposition in (9), as just pointed out, no longer is direct. Nevertheless, we may easily transfer the idea of defect groups of idempotents in $\bar{A}$ to those in A, which we proceed to do:

Lemma 1.13. Let $H \leq G$, let $\bar{e} \in \bar{A}^H$ be an arbitrary primitive idempotent, $e \in A^H$ a corresponding idempotent by Theorem I.11.2i). Let $K \leq H$. Then $\bar{e} \in \bar{A}_K^H$ if and only if $e \in A_K^H$.

Proof: Obviously, $\bar{e} \in \bar{A}_K^H$ if $e \in A_K^H$, as e maps to e by the canonical map. Conversely, if $\bar{e} \in \bar{A}_K^H$, then $e \in A_K^H + A^H\pi$, and thus $e \in A_K^H$ by Rosenberg's Lemma II.3.9, as obviously $e \notin A^H\pi$

We may therefore extend Definition 1.5.

Definition 1.14. Same notation as in Lemma 1.12. The defect groups of e are defined as the defect groups of $\bar{e}$.

Let us now see how this relates to projectivity.

Lemma 1.15. (Scott (1973)) Let θ equal F or R, and let $M \in \mathbf{M}_\theta(G)$ be an indecomposable trivial source module, say $M | I_H^{\uparrow G}$ where $H \leq G$ and I_H is the trivial $\theta[H]$-module. Set $A = (I_H^{\uparrow G}, I_H^{\uparrow G})$ and let $e \in A^G$ be an idempotent corresponding to M (cf. Theorem I.1.4). Let D be a defect group of e. Then D is a vertex of M.

Proof: As M is a direct summand of $I_H^{\uparrow G}$, $e \in A_K^G$ if and only if $(M,M)^G = (M,M)_K^G$ which by Corollary II.2.4 is equivalent to M being K-projective.

Lemma 1.16. (Green (1962a)) Let θ equal F or R, and let $M \in \mathbf{M}_\theta(G)$ be arbitrary. Let $e \in \theta[G]^H$ be a primitive idempotent for some $H \leq G$. Let $P \leq H$ be a p-group with $e \in (\theta[G])_P^H$. Then Me is P-projective.

Proof: Choose $a \in \theta[G]^P$ with $e = Tr_P^H(a)$. Then $m \to me$ induces the identity on Me, while $m \to ma$ is an $\theta[P]$-homomorphism. Thus this result follows from Corollary II.2.4 as well.

Theorem I.17. Let e be a block idempotent of $F[G]$. Then e is of defect 0 if and only if $eF[G]$ is a simple algebra.

Proof: If e is of defect 0, then any $eF[G]$-module is projective by Lemma 1.16, and thus $eF[G]$ is a simple algebra. Conversely, assume $eF[G]$ is a simple algebra and let E be the simple module of $eF[G]$. Let G act on $F[G]$ by conjugation. Then $eF[G] = (E,E) \simeq E \otimes E^*$, which is projective. Hence $Z(eF[G]) \simeq (E \otimes E^*)^G \simeq (E \otimes E^*)_1^G$, and thus any element in $Z(eF[G])$, including e is of the form $\sum_{g \in G}(\gamma)g = \sum_{g \in G} g^{-1}\gamma g$ for some $\gamma \in eF[G]$, which proves that e is of defect 0.

Remark: We recall that the blocks of $F[G]$ which are simple algebras, are completely determined by the character table of G if F is a splitting field, as discussed in Corollary I.16.8 and Proposition I.16.1.

2. Brauer's First Main Theorem.

We continue with the notation and assumptions of the previous section. Only, this time we work solely with coefficients in F. As most of our results deal with idempotents, we automatically have the corresponding results in characteristic 0.

Lemma 2.1. Let $P \leq G$ be a p-group, set $N = N_G(P)$ and let $a \in A^P$ be arbitrary. Then

(1) $$Br_P(Tr_P^G(a)) = Tr_P^N(Br_P(a)).$$

In particular, $Br_P(Tr_P^N(a)) = Tr_P^N(Br_P(a))$ and

(2) $$Br_P(A_P^G) = (F\Delta^P)_P^N .$$

Proof: By Proposition 1.1.i) and Lemma 1.10, $Br_P(Tr_P^G(a)) = Br_P(Tr_P^N(a))$, as $Tr_R^N((a)g)$, where $R = g^{-1}Pg \cap N$, is in the kernel of Br_P if $g \notin N$. However, as Δ^P and $\Delta\setminus\Delta^P$ are both N-invariant, it follows that $Br_P(Tr_P^N(a)) = Tr_P^N(Br_P(a))$.

Lemma 2.2. Let $P \Delta G$, P a p-group, and let $e \in A^G \cap F\Delta^P$ be a primitive idempotent with D as defect group. Then $P \leq D$.

Proof: By Lemma 1.7.

We now need the following well-known result for artinian algebras (or rings). Let A be an artinian algebra, e, f primitive idempotents. We then say that e and f are conjugate if there exists a unit $u \in A$ such that $e = u^{-1}fu$. Thus e and f are conjugate if and only if $eA \simeq fA$, as we saw in Theorem 1.3.12. Now, if furthermore B is an artinian algebra and $\phi : A \to B$ is a surjective homomorphism, then either $\phi(e) = 0$ or $\phi(e)$ is primitive, and ϕ induces a one-to-one correspondence between conjugacy classes of primitive idempotents of A not in the kernel of ϕ and conjugacy classes of primitive idempotents of B. Indeed, by Theorem I.3.14 we may assume that A and B are semi-simple, in which case the observation is immediate.

We may now prove the first main result.

Theorem 2.3. Let $P \leq G$ be a p-group and set $N = N_G(P)$. The Brauer map

(3) $$Br_P : A_P^G \to (F\Delta^P)_P^N$$

induces a bijection between conjugacy classes of primitive idempotents in A_P^G with defect group P in G and conjugacy classes of primitive idempotents in $(F\Delta^P)_P^N$ with defect group P in N.

Proof: We have seen that Br_P is a surjective algebra homomorphism from A_P^G onto $(F\Delta^P)_P^N$. Let $e \in A_P^C$ be a primitive idempotent with D as defect group in G. Then $D \underset{G}{\leq} P$ by Lemma 1.3 and thus $Br_P(e) \neq 0$ if and only if $D \underset{G}{=} P$ by Corollary 1.12. Thus the result follows from the classical result above, once we show that if $Br_P(e) = f \neq 0$, then f has defect group P in N. However, by definition of Br_P and Lemma 1.6, the defect group of f is contained in that of e, i.e. in P, as $P \Delta N$, while the other inclusion follows from Lemma 2.2.

In the case where A is the endomorphism ring of a permutation module, this is Theorem 3 of Scott (1973). In the case where $A = F[G]$, this is

Brauer's First Main Theorem 2.4. (Brauer (1956)) Let $P \leq G$ be a p-group, and set $N = N_G(P)$, $C = C_G(P)$. Then the Brauer map

(4) $$Br_P : (F[G])_P^G \to (F[C])_P^N$$

induces a bijection between block idempotents of F[G] with P as defect group in G and block idempotents of F[N] with defect group P in N.

In particular, we have implicitely stated that

Lemma 2.5. Let $P \triangle G$ be a P-group and let e be a block idempotent of F[G], with D as defect group in G. Then $e \in (F[C_G(P)])^G$. In particular, $P \leq D$.

Proof: Set $A = F[G]$. By Lemma 1.8,

(5) $$Z(A) = (F[C_G(P)])^G \oplus \left(\sum_{R < P} A_R^P \cap Z(A)\right) .$$

Also, as P acts trivially on any simple A-module and $|P : R|$ is a power of p for $R < P$, $A_R^P \subseteq J(A)$ and thus

(6) $$Z(A) \subseteq (F[C_G(P)])^G + J(Z(A)) .$$

As e is an idempotent, this forces $e \in (F[C_G(P)])^G$. Thus $D \leq P$ by Lemma 1.7.

As $F[G]^G = Z(F[G])$, Theorem 2.4 follows.

Brauer's First Main Theorem reduces the problem of determining which blocks of the group algebra F[G] have a certain p-subgroup D as defect group to the similar problem in $F[N_G(D)]$. We therefore proceed to investigate this problem in the following section.

It will be of interest to take a closer look at the restriction of the Brauer map Br_P to the center of any block B for an arbitrary but fixed p-subgroup P of G. Denote the block idempotent corresponding to B by $\underline{B}$. We have already seen that $Br_P(Z(B)) = 0$ unless P is a subgroup of a defect group of B.

Notation. Let $\mathcal{K}_1, \ldots, \mathcal{K}_k$ denote the conjugacy classes of G. Recall that for $A = F[G]$, A_P^G is spanned by $\{[\mathcal{K}_i] \mid D(\mathcal{K}_i) \leq_G P\}$.

Set $A^G_{<P} = \sum_{R<P} A^G_R$ and let $A^G_{=P}$ be spanned by $\{[\mathcal{K}_i] \mid D(\mathcal{K}_i) =_G P\}$. Thus

(7) $$A^G_P = A^G_{<P} \oplus A^G_{=P} .$$

In particular,

(8) $$A^G_P \underline{B} = A^G_{<P}\underline{B} + A^G_{=P}\underline{B} .$$

The following result goes back to Rosenberg (1961).

Lemma 2.6. With the notation above,

i) $\text{Ker } Br_P \cap A^G_P\underline{B} = A^G_{<P}\underline{B}$

ii) $A^G_P\underline{B}/A^G_{<P}\underline{B} \simeq Br_P(A^G_{=P}\underline{B}) = (F[C_G(P)]_P^{N_G(P)}\ Br_P(\underline{B})$

$$= \bigoplus_i (F[C_G(P)]_P^{N_G(P)}\ \underline{b}_i$$

where the sum is over those blocks b_i of $F[C_G(P)]$ for which $\underline{b}_i Br_P(\underline{B}) \neq 0$.

iii) $Br_D(Z(B)) = (F[C_G(D)])_D^{N_G(D)}\ \tilde{\underline{B}}$, where D is a defect group of B and $\tilde{B}$ is the block of $F[N_G(D)]$ corresponding to B.

Proof: By Lemma 1.10, $\text{Ker } Br_P \cap A^G_P = A^G_{<P}$. As $A^G_{<P} \cap B = A^G_{<P}\underline{B}$, i) follows

The (first) isomorphism of ii) is by i) and (8). As $Br_P : A^P \to F[C_G(P)]$ is a homomorphism, the first equality of ii) follows from Lemma 2.1, and the second simply from the fact that $Br_P(\underline{B}) = \Sigma \underline{b}_i$ with the notation above, as $Br_P(A^G_P\underline{B})$ is an ideal.

Definition 2.7. Broué (1979), Olsson (1980) The dimension over F of $A^G_P\underline{B}/A^G_{<P}\underline{B}$, or $Br_P(A^G_{=P}\underline{B})$, is called the multiplicity of P as a lower defect group of B. We will return to this in Section 10.

Brauer's original definition of the Brauer homomorphism was the following. Let $P \leq G$ be a p-group and let $PC_G(P) \leq H \leq G$. We then define $Br_P : Z(F[G]) \to Z(F[H])$ by setting $Br_P([\mathcal{K}]) = [\mathcal{K}] \cap C_G(P)$ for any G-conjugacy class $\mathcal{K}$. Thus the Brauer homomorphism is simply the restriction of the Brauer map to $Z(F[G])$, and it follows from Corollary 1.10 that this map indeed is a homomorphism.

So a natural question arises: Why can we replace $Z(F[H])$ with $(F[C])^N$ and still get Brauer's First Main Theorem? The answer is provided by

<u>Osima's Theorem 2.8</u>. (See Osima (1955) Let $e \in F[G]$ be a central idempotent. Then e is a linear combination of p-regular elements.

<u>Proof</u>: (Passman (1969)) We may as well assume that e is primitive as a central idempotent. Let $e = \sum_{g \in G} \alpha_g g$ and assume $\alpha_g \neq 0$ for some g. Let $g = g_p g'$ be the p-decomposition of g and assume $g_p \neq 1$. Let $D = \langle g_p \rangle$. Then $Br_D(e) \neq 0$ is a central idempotent of $F[C_G(D)]$. So it suffices to prove the theorem for $G = C_G(D)$.

Let $|G| = p^a m$ where $(p,m) = 1$ and set $q = ord(g')$. Then there exists $n \geq a$ such that $p^n \equiv 1 \bmod q$. Now

(9) $$(g'^{-1}e)^{p^n} = (g'^{-1})^{p^n} e = g'^{-1}e \ .$$

However, recall from Lemma I.13.4 that if $S(F[G])$ denotes the vector space in $F[G]$ spanned by elements of the form $xy - yx$ for $x, y \in G$, then

(10) $$(g'^{-1}e)^{p^n} = (\sum_{x \in G} \alpha_x g'^{-1}x)^{p^n} \equiv \sum_{x \in G} \alpha_x^{p^n} (g'^{-1}x)^{p^n}$$

mod $S(F[G])$. But $(g'^{-1}x)^{p^n}$ is a p-regular element for all x, while (9) shows the coefficient of g_p in $g'^{-1}e$ is $\alpha_g \neq 0$. Consequently, there exists $h, k \in G$ with $g_p = hk$ and $hk - kh \neq 0$. However, as $g_p \in Z(G)$, $g_p = h^{-1}g_p h = kh$ as well, a contradiction. Thus $g_p = 1$ and the theorem follows.

It is not entirely clear off-hand that a similar result holds in $R[G]$. However, it suffices of course to handle the case where S is a splitting field of $S[G]$ and this is easy to see directly:

<u>Theorem 2.9</u>. (Osima (1955)) Let S be a splitting field of $S[G]$ and let $\mathbb{B}$ be a p-block of G. Denote the irreducible Brauer characters of $\mathbb{B}$ by $\phi_1,\ldots,\phi_{\ell_0}$ and the characters of the corresponding

p.i.m.'s by $\Phi_1,\dots,\Phi_{\ell_0}$. Let e be the corresponding block idempotent of $R[G]$. Then

$$(11)\qquad e = \frac{1}{|G|}\sum_{g\in G}\Big(\sum_{i=1}^{\ell_0}\phi_i(1)\Phi_i(g^{-1})\Big)g = \frac{1}{|G|}\sum_{g\in G_0}\Big(\sum_{i=1}^{\ell_0}\Phi_i(1)\phi_i(g^{-1})\Big)g.$$

Proof: Let $\chi_1,\dots,\chi_{k_0}$ be the irreducible ordinary characters of $\mathbb{B}$, and let e_1 be the unity of the Wedderburn component of $S[G]$ corresponding to χ_i . Then

$$(12)\qquad e = \sum_{i=1}^{k_0} e_i = \frac{1}{|G|}\sum_{g\in G}\sum_{i=1}^{k_0}\chi_i(1)\chi_i(g^{-1})g$$

and (11) follows from this and Lemma I.15.7.

3. Blocks of groups with a normal subgroup.

This section is mainly inspired from Alperin and Broué (1979) and Landrock (1981c). Notice however again that some of the results are much older. To prove Theorem 3.5 we use an idea of Külshammer (1980).

Throughout the section we will assume that $1 \neq H$ is a normal subgroup of G and later we will furthermore assume that $H = P$ is a p-group.

Notation: Whenever B is a block of some algebra, the corresponding block idempotent will be denoted by $\underline{B}$.

Definition 3.1. Let b_H be a block of $F[H]$. By the normalizer or inertial group $N(b_H) = N_G(b_H)$ of b_H in G we mean the subgroup of G fixing $\underline{b_H}$ by conjugation.

Lemma 3.2. Let b_H be a block of $F[H]$. Then

i) $\underline{b_H}$ is a central idempotent of $F[N(b_H)]$.

ii) $\mathrm{Tr}^G_{N(b_H)}(\underline{b_H})$ is a central idempotent of $F[G]$.

Assume furthermore that $H = P$ is a p-group and let b be an arbitrary block of $F[G]$. Then

iii) $\underline{b} \in (F[C_G(P)])^G$.

iv) Let D be a defect group of b in G. Then $P \leq D$.

v) $\underline{b}$ is of the form given in ii) for b_H satisfying $\underline{b}\,\underline{b}_H \neq 0$.

vi) $\underline{b}_H$ is a block idempotent of $F[N(b_H)]$.

Proof: i) and ii) are trivial, iii) and iv) have already been proved in Section 2, v) follows from ii) and iii). Finally, that $\underline{b}_H$ is a primitive idempotent in $F[N(b_H)]$ follows from the fact that any block idempotent of $F[N(b_H)]$ lies in $F[C_G(P)]$ by iii) where it is primitive as such.

The connection between the algebra structure of b_H and that of $\mathrm{Tr}_{N(b_H)}^{G}(\underline{b}_H)F[G]$ is easy to describe:

Lemma 3.3. (Külshammer (1980)) Let θ equal F or R, let b_H be a block of $\theta[H]$ and let b be the sum of blocks of $\theta[G]$ such that the unity of b is $\underline{b} = \mathrm{Tr}_{N(b_H)}^{G}(\underline{b}_H)$. Let $s = |G : N(b_H)|$. Then

(1) $$b \simeq \mathrm{Mat}_s(\underline{b}_H\theta[N(b_H)]).$$

Proof: Set $A = \underline{b}_H\theta[N(b_H)]$. Then $\theta[G]\underline{b}_H$ is a right A-module and free as such of dimension s. Thus

(2) $$(\theta[G]\underline{b}_H,\theta[G]\underline{b}_H)^A = \mathrm{Mat}_s(A)$$

as $A \simeq (A,A)^A$. However b maps into the former by left multiplication, and this map is an injective homomorphism. Indeed, if $\underline{b}\, x\, y\underline{b}_H = 0$ for all $y \in \theta[G]$ and some $x \in \theta[G]$, then $\underline{b}\, x\, \underline{b} = \underline{b}x = 0$, as $\underline{b} = \mathrm{Tr}_{N(b_H)}^{G}(\underline{b}_H)$. Finally, it is onto as $(1-\underline{b})\theta[G]\underline{b}_H = 0$ and $\theta[G]\underline{b}_H$ is a direct summand of $\theta[G]$ as a left ideal.

Definition 3.3. Same notation as in Lemma 3.3. Let $\mathrm{Tr}_{N(b_H)}^{G}(\underline{b}_H) = \Sigma \underline{B}_i$, where B_i runs through blocks of $\theta[G]$. Then B_i is said to cover b_H.

As a consequence of Lemma 3.3, we get

Theorem 3.4. Same notation as in Lemma 3.2.

There is a one-to-one correspondence between blocks $\tilde{B}_i$ of $\theta[N(b_H)]$ covering b_H in $N(b_H)$ and blocks B_i of $\theta[G]$ covering b_H in G, given by

(3) $$\underline{B}_i = \mathrm{Tr}_{N(b_H)}^{G}(\tilde{\underline{B}}_i)$$

(4) $$B_i \simeq \mathrm{Mat}_s(\tilde{B}_i)$$

where $s = |G : N(b_H)|$.

Proof: By (1), and the fact that blocks are orthogonal,

(5) $$\oplus B_j \cong \mathrm{Mat}_s(\oplus \tilde{B}_i) \simeq \oplus \mathrm{Mat}_s(\tilde{B}_i)$$

where we sum of all blocks in G respectively $N(b_H)$ which cover b_H. As $\mathrm{Mat}_s(\tilde{B}_i)$ is indecomposable, the claim follows from Theorem I.4.4.

Theorem 3.5. (Fong (1961), Reynolds (1963)) Let b_H be a block of $\theta[H]$ and let $\tilde{B}$ resp. B be corresponding blocks of $F[N(b_H)]$ resp. $\theta[G]$ which covers b_H. Denote the corresponding p-blocks of $N(b_H)$ resp. G by $\tilde{\mathbb{B}}$ resp. $\mathbb{B}$. Let $K = F$ or S according to whether θ equals F or R. Then

i) Let N be a simple $K[N(b_H)]$-module in $\mathbb{B}$. Then $N^{\uparrow G}$ is a simple module in $\mathbb{B}$ and this sets up a one-to-one correspondence between the sets of simple modules in $\tilde{\mathbb{B}}$ resp. $\mathbb{B}$.

ii) $\tilde{\mathbb{B}}$ and $\mathbb{B}$ have identical decomposition matrix and Cartan matrix w.r.t. this correspondence.

iii) $\tilde{B}$ and B have a defect group in common. In particular, $N(b_H)$ contains a defect group of B.

Proof: Let M be a simple B-module. Then $M\tilde{\underline{B}} \neq 0$ by Theorem 3.4 and consequently $M\underline{b}_H \neq 0$. Let X be a simple $K[N(b_H)]$-submodule of $M\underline{b}_H$. Then obviously the inertial group T of X is contained in $N(b_H)$, and it follows from Theorem II.11.1 that $M = N^{\uparrow G}$ for some $K[N(b_H)]$-submodule N of $M\underline{b}_H$ containing X. Hence N is simple as well and N lies in $\tilde{\mathbb{B}}$. Conversely, if N is a simple

$K[N(b_H)]$-module in $\tilde{\mathbb{B}}$, we let M be a simple submodule of $N^{\uparrow G}$. Then again M lies in $\mathbb{B}$ by Theorem 3.4 and the argument above shows that in fact $M = N^{\uparrow G}$. Finally, as B and $\tilde{B}$ have the same number of isomorphism classes of simple modules by Theorem 3.4, i) is proved.

ii) is an immediate consequence of i).

iii) By Theorem 3.4, (3), a defect group D of B is contained in some defect group $\tilde{D}$ of $\tilde{B}$. As $\underline{B}\underline{\tilde{B}} = \underline{\tilde{B}}$, Proposition 1.1 together with Lemma 1.4 yield equality.

As we do not wish to discuss the structure of group algebras over p-solvable groups in particular but refer to Feit (1982) or Blackburn and Huppert (1982) for this we restrict ourselves in the following to the case where $H = P$ is a p-group. We point out that the rest of this section only uses Lemma 3.2.

Theorem 3.6. Let $1 \neq P$ be a normal p-subgroup of G, and let b be a block of $F[G]$. By Lemma 3.2, let b_P be a block of $F[C_G(P)]$ such that $\underline{b_P}\underline{b} \neq 0$. Then

i) (Landrock (1981c)) Let $Q \in \mathrm{Syl}_p(N(b_P))$. Then the defect groups of b in G are the G-conjugates of the defect groups of b_P in $C_G(P)Q$.

ii) Let D_N be a defect group of b_P in $N(b_P)$. Then

a) D_N is a defect group of b in G.

b) $D_C = D_N \cap C_G(P)$ is a defect group of b_P in $PC_G(P)$.

iii) (Olsson) $Z(D_N) \leq D_P$.

Proof: ii) a) follows from Theorem 3.5 iii) but we will give a short direct proof: Let D be a defect group of b in G. Then $D_N \underset{G}{\leq} D$ by Lemma 1.7 while Lemma 3.2 v) asserts that the other inclusion holds.

Next we let D_Q be a defect group of b_P in $C_G(P)Q$. Again, assume $D_Q \leq D_N$ by Lemma 1.7. Moreover, we may assume that $D_N \leq C_G(P)Q$ by choice of Q. We must show that $D_N \leq D_Q$. Now let $\underline{b_P} = \Sigma\alpha_C[C]$, sum over $C_G(P)$-conjugacy classes in $C_G(P)$. Let C_0 be one with $\alpha_{C_0} \neq 0$ such that $D(C_2) = N(b_P)\ D_N$, where C_2 is the $N(b_P)$-conjugacy class containing C_0, again by Lemma 1.6. Hence the

$C_G(P)Q$-conjugacy class C_1 containing C_0 has D_N as defect group as well. Furthermore, as $\underline{b}_P$ is central in $F[C_G(P)Q]$, the coefficient of $[C_1]$ in the expression for $\underline{b}_P$ in $F[C_G(P)Q]$ is α_{C_0}, and thus $D_N \underset{N(\overline{b}_P)}{\leq} D_Q$ as claimed, which proves i).

Again, Lemma 1.6 yields that a defect group of b_P in $C_G(P)$ is $N(b_P)$-conjugate to a subgroup of D_N. Thus it may be picked as a subgroup of D_C. But then Lemma 1.7 in fact guarantees equality, and ii), b) is proved.

Lemma 3.2 v) and b) yield that PD_C is contained in some defect group of b_P in $PC_G(P)$. On the other hand

$$\underline{b}_P \in (F[C_G(P)])_{D_C}^{C_G(P)} \subseteq (F[PC_G(P)])_{PD_C}^{PC_G(P)} \tag{6}$$

and thus PD_C is a defect group of b_P in $PC_G(P)$.

iii) follows immediately from ii) b) and c).

<u>Corollary 3.7.</u> (Brauer (1956), Hamernik and Michler (1972)) Same notation and assumption as in Theorem 3.6. Assume furthermore that $|N(b_P) : PC_G(P)|$ is prime to p. Then $D \underset{G}{=} D_P$.

In the following we set $\overline{G} = G/P$ and denote by $\tau : F[G] \to F[\overline{G}]$ the algebra homomorphism induced by the canonical homomorphism $G \to \overline{G}$.

<u>Lemma 3.8.</u> Same notation as above. Then

i) Ker τ is a nilpotent ideal in $F[G]$.

ii) Let $\mathcal{K}$ be a conjugacy class in G such that $\mathcal{K} \cap C_G(P) = \emptyset$. Then $[\mathcal{K}] \in \operatorname{Ker} \tau$.

<u>Proof:</u> i) Let A be the augmentation ideal of $F[P]$. Then $\operatorname{Ker} \tau = F[G]A = AF[G]$ (cf. Lemma I.11.15), and thus $(\operatorname{Ker} \tau)^i = F[G]A^i$. But $A = J(P)$, as P is a p-group, and thus A is nilpotent.

ii) If $\mathcal{K} \cap C_G(P) = \emptyset$, then any orbit X of P on $\mathcal{K}$ has length a power of p. However, if $x \in X$, then $\tau([X]) = |X|x = 0$ where $[X] = \sum_{y \in X} y$ and thus $\tau([\mathcal{K}]) = 0$.

In particular, there is a one-to-one correspondence between primitive idempotents of $F[G]$ and those of $F[\overline{G}]$. However, it is not always true that there is a one-to-one correspondence between central idempotents. In order to ensure this, we need an additional assumption, but an assumption we fortunately can afford by Lemma 2.5 and Theorem 3.6.

Lemma 3.9. Same notation as above. Assume furthermore that $G = PC_G(P)$. Then τ induces a one-to-one correspondence between block idempotents of $F[G]$ and those of $F[\overline{G}]$.

Proof: We shall give a straightforward module theoretic proof of this statement (See Landrock (1981c).). Let b resp. $\{\overline{b}_i\}_1^r$ be blocks of $F[G]$ resp. $F[\overline{G}]$ such that $\tau(\underline{b}) = \sum_{t=1}^{r} \overline{\underline{b}}_i$. If $r > 1$, then by our initial definition of blocks in Chapter I, Section 4, there exists simple b_{i_j}-modules W_{i_j}, $j=1,2$ for suitable $i_j \leq r$ and an indecomposable $F[G]$-module W satisfying

(7) $$0 \to W_{i_1} \to W \to W_{i_2} \to 0.$$

Let $z \in Z(P)$ and consider the $F[G]$-endomorphism $\phi_z : w \to w(1-z)$ of W. As z is a p-element $\mathrm{Ker}_{\phi_z} \neq 0$. Thus Ker_{ϕ_z} equals W_{i_1} or all of W. But in the former case, $W_{i_2} \simeq \mathrm{Im}\phi_z \subset W$, a contradiction, as W_{i_1} is the only non-trivial submodule of W. Thus $Z(P)$ acts trivially on W. Using induction, we therefore obtain that P acts trivially on W. Hence W is an $F[\overline{G}]$-module, a contradiction.

Remark. Notice that this proof just uses the basis properties of blocks.

Using the same technique, we may also prove

Lemma 3.10. Same notation and assumption as in Lemma 3.9. Denote the Cartan matrix of a group algebra $F[H]$ by $\underline{\underline{C}}_H$. Then

(8) $$\underline{\underline{C}}_G = |P|\underline{\underline{C}}_{\overline{G}} .$$

Proof: Using induction on $|P|$ it suffices to consider the case where $P = \langle x \rangle$ is cyclic of order p. Let ϕ_i denote the $F[G]$-endomorphism which maps a to $a(1-x)^i$. The kernel of ϕ_{p-1} is obviously $J(F[\langle x \rangle])F[G]$, and thus

(9) $$F[G](1-x)^{p-1} \simeq F[\bar{G}].$$

In particular, if A is an indecomposable projective $F[G]$-module, then $A(1-x)^{p-1}$ is the corresponding projective indecomposable $F[\bar{G}]$-module. Moreover,

(10) $$A \supset A(1-x) \supset A(1-x)^2 \supset \ldots \supset A(1-x)^{p-1}$$

is a filtration, where

(11) $$A(1-x)^i/A(1-x)^{i+1} \simeq P(1-x)^{p-1}$$

for all i, from which (8) follows.

Lemma 3.11. Let b be a block of $F[G]$, where $G = PC_G(P)$, moreover P a p-group, with $D \geq P$ as a defect group in G. Then the corresponding block $\bar{b}$ of G/P has D/P as a defect group in G/P.

Proof: Let τ be as above. As $\tau(\underline{b}) = \underline{\bar{b}}$, Lemma 1.4 asserts that if $\bar{D}$ is a defect group of $\underline{\bar{b}}$, then $\bar{D} \underset{G/P}{\leq} D/P$, while the other inclusion follows from Lemma 1.7.

Corollary 3.12. Assume $D = P$ in Lemma 3.11. Then $\bar{b}$ is of defect 0. In particular, $\bar{b}$ is a simple algebra and thus $\bar{b}$ has only one simple module up to isomorphism.

Proof: By Lemma 3.11, $\bar{b}$ is of defect 0. Thus any $\bar{b}$-module is projective by lemma 1.16, and the rest follows from Proposition I.9.1.

We are now ready to prove the socalled Extended First Main Theorem which shows how defect groups may easily be determined from the local structure. Our proof of the First Main Theorem showed a one-to-one correspondence between block idempotents of $F[G]$ with D as defect group in G and $N_G(D)$-conjugacy classes of block idempotents in $F[C_G(D)]$ with D as defect group in $N_G(D)$. Moreover, as we have just seen, there is a one-to-one correspondence between block idempotents of $F[C_G(D)]$ with D as defect group in $DC_G(D)$ and block idempotents of $F[DC_G(D)/D]$ of defect

0. Thus we are left with the following problems:

1) When does a block b_D of $F[C_G(D)]$ with D as defect group in $DC_G(D)$ have defect group D in $QC_G(D)$ where $Q \in Syl_p(N(b_D))$?

2) How does one determine the blocks of defect 0?

4. The Extended First Main Theorem.

We now answer the first question raised in Section 3, following Landrock (1981c).

The Extended First Main Theorem 4.1. Let G be a finite group, $D \leq G$ a p-group. For each block b of $F[C_G(D)]$ with defect group D in $DC_G(D)$, let U_b denote the unique simple module of b by Corollary 3.9 and let Q_b be a Sylow p-subgroup of $N(b)$.

The Brauer homomorphism induces a one-to-one correspondence between blocks of $F[G]$ with D as defect group in G and $N_G(D)$-conjugacy classes of blocks b of $F[C_G(D)]$ with D as defect group in $DC_G(D)$ and $U_b^{\uparrow Q_bC_G(D)}$ semi-simple.

Before we prove this, we derive Brauer's original version as a corollary.

Corollary 4.2. Let $D \leq G$ be a p-group and let b be a block of $F[C_G(D)]$ with D as defect group in $DC_G(D)$. Then

i) (Brauer (1956), Hamernik and Michler (1972)) Assume $|N(b) : DC_G(D)|$ is prime to p. Then the corresponding block of $F[N_G(D)]$ (in view of Lemma 3.2) has defect group D in $N_G(D)$.

ii) (Brauer (1956)) Assume F is a splitting field of $bF[C_G(D)]$, and assume furthermore that the corresponding block of b in $F[N_G(b)]$ has D as defect group in $N_G(D)$. Then $|N(b) : DC_G(D)|$ is prime to p.

Proof: Let U denote the simple module of $bF[DC\ (D)]$.

i) has already been observed (Corollary 3.3). However, if $|N(b) : DC_G(D)|$ is prime to p, then certainly $U^{\uparrow N(b)}$ is semisimple, as we saw in Theorem II.11.2 i), so i) is also an immediate corollary of our theorem.

ii) As U is in an $F[DC_G(D)/D]$-block of defect 0, U is liftable by Lemma 1.16 and the fact that projective modules are liftable. Moreover, as F is a splitting field of $bF[C_G(D)]$, $\dim_F U$ divides $|C_G(D)|$. In fact $\dim_F U = |\overline{P}|h$, where $P \in Syl_p(DC_G(D))$, $\overline{P} = P/D$ and h is prime to p. Let $P \leq Q \in Syl_p(N(b))$, and set $H = QC_G(D)$, $\overline{H} = H/D$. Then $\dim_F(U^{\uparrow\overline{H}}) = |Q/D|h$. As any indecomposable component of $U^{\uparrow\overline{H}}$ has dimension a multiple of $\dim_F(U)$, and of $|Q/D|$ as well, $U^{\uparrow\overline{H}}$ is indecomposable. (We could, of course, have quoted Green's Theorem II.11.10.) But obviously, $U^{\uparrow\overline{H}}$ lies in the corresponding block in $F[\overline{H}]$ and thus is simple by Theorem 4.1. However, as U is the only simple module of $\underline{\overline{b}}F[DC_G(D)/D]$, $U \otimes \overline{x} \simeq U$ for all $\overline{x} \in \overline{H}$ and thus

(1) $$(U^{\uparrow\overline{H}}, U^{\uparrow\overline{H}})^{\overline{H}} \simeq (U^{\uparrow\overline{H}}{}_{\downarrow\overline{C}}, U)^{\overline{C}} ,$$

where $\overline{C} = DC_G(D)/D$ is of dimension $|\overline{H} : \overline{C}|$. Hence $H = DC_G(D)$, and ii) is proved.

In view of our discussion in Section 3, Theorem 4.1 is an immediate consequence of

Lemma 4.3. Let D be a normal p-subgroup of G. Set $\overline{G} = G/D$, and let b be a block of $C_G(D)$ with defect group D in $DC_G(D)$. Let $\overline{b}$ be the corresponding block of $F[DC_G(D)/D]$.

Assume $G = N(b)$. Then the following are equivalent.

i) $\underline{b}$ has defect group D in G.

ii) $\underline{\overline{b}} \in (F[\overline{G}])_{\overline{1}}^{\overline{G}}$.

iii) $\underline{\overline{b}}F[\overline{G}]$ is a semisimple algebra.

iv) Let U be the simple module in $\underline{b}F[DC_G(D)]$. Then $U^{\uparrow G}$ is semisimple.

Proof: i) and ii) are equivalent by Lemma 3.8 and the definition of a defect group, and ii) and iii) are equivalent by Theorem 1.17. Finally, as $\overline{b}$ is $\overline{G}$-invariant,

(2) $$\underline{\overline{b}}F[\overline{G}] \simeq \underline{\overline{b}}(F[\overline{C}])\uparrow_{\overline{C}}^{\overline{G}} = (\underline{\overline{b}}F[\overline{C}])\uparrow_{\overline{C}}^{\overline{G}}$$

and $\underline{\overline{b}}F[\overline{C}] \simeq U^{(n)}$ for some $n \in \mathbb{N}$, as an $F[\overline{C}]$-module, where as before

$\overline{C} = DC_G(D)/D$. Hence $\overline{b}F[G]$ is semisimple if and only if $U^{\uparrow\overline{G}}$, or equivalently, $U^{\uparrow G}$ is semisimple, as D is in the kernel of U.

Example 1. To illustrate the subtleties of the general extended first main theorem, consider a group $G = CW$, semidirect product, with C normal, $C = D \times K$ where $D \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $K \simeq \mathbf{Z}_3$, and $W \simeq \mathbf{Z}_2$. Assume furthermore that W acts non-trivially on D. Finally, F is a field of characteristic 2. Note that $GF(4)$ is a splitting field of K.

Case i) $KW \simeq \Sigma_3$, $F = GF(4)$. Then $C = C_G(D)$ has two nonprincipal blocks b_1 and b_2 with defect groups D, and $N(b_i) = C$. It follows that G has a block $b = b_1 + b_2$ with defect group D. Correspondingly $|N(b_i) : C| = 1$.

Case ii) $KW \simeq \Sigma_3$, $F = GF(2)$. Here, C only has one non-principal block b. Let U be the simple module of b. As $N(b) = G$, G is the inertial group of U, and W acts on $F[K] \simeq I \oplus U$, where I is the trivial $F[K]$-module. Thus U may be extended to an $F[G]$-module U', and $U'_{\downarrow W}$ is projective, as KW is non-abelian. Consequently, $U^{\uparrow G} \simeq U' \oplus U'$ is semisimple, so b is a block of $F[G]$ with defect group D in spite of the fact that $|N(b) : C| = 2$.

Case iii) $KW \simeq \mathbf{Z}_6$, $F = GF(4)$. As $O_2(G) = DW$, neither b_1 nor b_2, with the notation of Case i), has defect group D. If U_i is the simple module of b_i then $U_1^{\uparrow G}$ has Loewy series $\begin{matrix} U_2 \\ U_1 \end{matrix}$ and thus is not semisimple. Correspondingly, $|N(b_i) : C| = 2$.

Case iv) $KW \simeq \mathbf{Z}_6$, $F = GF(2)$. Again $O_2(G) = DW$ so b, with the notation of ii) does not have defect group D, and the simple module $F[C]$-module U in b extends to an $F[G]$-module U', but this time $U'_{\downarrow W}$ is trivial. Correspondingly, $U^{\uparrow G}$ has Loewy series $\begin{matrix} U' \\ U' \end{matrix}$, whence is not semisimple. Again, $|N(b) : C| = 2$, but this does not tell us anything.

5. Defect groups and vertices.

Our next aim is to find some sort of characterizations of blocks in terms of their defect groups. We have already seen an example of this in Theorem 1.17. A key is provided by the following fairly straightforward but important result due to Nagao (1963), which will lead us to a proof of Brauer's Second Main Theorem.

We use the notation of the previous sections. Recall that for e a block idempotent of $F[G]$ and $P \leq G$ a p-group, $e_P := Br_P(e)$ is a central idempotent in $F[C_G(P)]$. Moreover, e lifts uniquely to a block idemopotent $\hat{e}$ of $R[G]$, and if $e_P = \Sigma f_i$, where the f_i's are block idempotents of $C_G(P)$, then all f_i's left uniquely to block idempotents $\hat{f}_i$ of $R[C_G(P)]$. Hence $\hat{e}_P$ lifts uniquely to e_P. So in any case if θ equals F or R and e is a block idempotent of $\theta[G]$, there corresponds uniquely a central idempotent e_P in $\theta[C_G(P)]$, determined as above.

<u>Theorem 5.1</u>. (Nagao) Let $P \leq G$ be a p-group and let H be any group with $C_G(P)P \leq H$. Let e be a block idempotent and define the central idempotent $e_H \in \theta[H]$ as the sum of all block idempotents f such that $f_P e_P \neq 0$, where f_P and e_P are described above. For any $M \in \mathbf{M}_\theta(G)$, set

(1) $$Me_{\downarrow H} = Me_H \oplus (\bigoplus_j N_j)$$

where N_j is indecomposable. Let V_i be a vertex of N_i. Then $P \not\leq V_i$.

<u>Proof</u>: Set $\varepsilon = e - ee_H$. If $\varepsilon = 0$, there is nothing to prove. Assume therefore that $\varepsilon \neq 0$. Then ε and e_H are orthogonal idempotents and consequently

(2) $$Me_{\downarrow H} = Me_H \oplus M\varepsilon$$

as an $\theta[H]$-module, as $\varepsilon \in (\theta[G])^H$. Let $\varepsilon = \sum_r \varepsilon_r$ be a primitive idempotent decomposition in $(\theta[G])^H$. By Lemmas 1.4 and 1.13, there exist p-subgroups Q_r of H such that $\varepsilon_r \in (\theta[G])^H_{Q_r}$. Moreover, $Br_P(\varepsilon_r) \in (\pi)[C_G(P)]$ by definition of Br_P and ε_r. But then the statement follows from Corollary 1.11 and Lemma 1.16.

As an application of this and Green Correspondence, we obtain

<u>Corollary 5.2</u>. (Green (1964)) Same notation as above. Assume furthermore that M is an indecomposable $e\theta[G]$-module with P as a

vertex. Then there exists an indecomposable $\theta[H]$-module N such that

i) $N | (M_{\downarrow H})e_H$

ii) $M | N^{\uparrow G}$.

Proof: Let $f(M)$ be the Green correspondent of M in $N_G(P)$. Then $f(M) = f(M)e_P$ by Theorem 5.1, applied to $N_G(P)$, and Lemma 2.5. Furthermore, if $f(M)_{\downarrow H \cap N_G(P)} = \bigoplus_i X_i$ with X_i indecomposable, X_i has vertex P. Let $g(X_i)$ be the Green correspondent in H. Then $g(X_i) | M_{\downarrow H}$ for some i, as $M_{\downarrow H}$ has an indecomposable direct summand with vertex P, again by Theorem 5.1. Moreover as $f(M) | X_i^{\uparrow N_G(P)}$, we also have that $M | g(X_i)^{\uparrow G}$. Finally, as $g(X_i)$ has P as vertex $g(X_i) = g(X_i)e_H$ by Theorem 5.1. Thus $g(X_i) = N$ will do.

We are now able to characterize defect groups in term of vertices, thereby generalizing Theorem 1.17.

Corollary 5.3. Let e be a block idempotent of $\theta[G]$ and let D be a defect group of e. Then all $e\theta[G]$-modules are D-projective, and there exists an indecomposable $\theta[G]$-module with D as vertex.

Proof: One way has already been proved in Lemma 1.16. To prove the other way, it suffices to consider the case where D is normal by Corollary 5.2 and Brauer's First Main Theorem. Let U be a simple $eF[G]$-module, where e is the corresponding (or same) block idempotent of $F[G]$. Let V be a vertex of U. Then $V \leq D$, and $U_{\downarrow D}$ is a trivial module. Let L be a source of U in $F[V]$. Then

(3) $$U_{\downarrow D} | (L^{\uparrow G})_{\downarrow D} = \sum_{x \in V\backslash G/D} (L \otimes x)_{\downarrow V^x \cap D}{}^{\uparrow D} .$$

But no direct summand of $((L \otimes x)_{\downarrow V^x \cap D})^{\uparrow D}$ is isomorphic to the trivial $F[D]$-module unless $V^x = D$, which forces $V = D$.

Furthermore, as U is a projective $F[G/D]$-module, U lifts uniquely to an $R[G/D]$-module, which is projective as well and therefore considered as an $R[C]$-module must have D as vertex by Lemma II.1.3. Thus both $\theta = F$ and $\theta = R$ are covered.

Corollary 5.4. Same notation as in Corollary 5.3. Let M be an $e\theta[G]$-module. Then $|P : D|$ divides $\mathrm{rank}_\theta(M)$, where $D \leq P \in \mathrm{Syl}_p(G)$.

Proof: By Green's Theorem, as we may assume F to be algebraically closed (see Appendix III).

In particular, if $\mathbb{B}$ is a p-block of G and S is a splitting field, let $\chi_1, \ldots, \chi_{k_0}$ be the irreducible characters of $\mathbb{B}$. Let $|G| = p^a m$ where $(p,m) = 1$, and let $\mathbb{B}$ be of defect d. Then p^{a-d} divides $\chi_i(1)$ for all i. We now introduce

Definition 5.5. Same notation as above. Let $\chi_i(1) = p^{a-d+h_i} m_i$, where $(p,m_i) = 1$. Then h_i is called the height of χ_i.

Similarly, we define the height of any generalized or Brauer character of $\mathbb{B}$.

Corollary 5.6. Let S be a splitting field of $S[G]$. Then any p-block of G contains an irreducible character of height 0.

Proof: Go back to the proof of Corollary 5.3. There the height of the character of U is 0, from which the statement easily follows.

Another consequence of Theorem 5.1 is the following. (See Juhász (1981), Thm. 2.)

Corollary 5.7. Same notation as in Theorem 5.1. Let N be an indecomposable $\theta[H]$-module with vertex P such that $N = Ne_H$.

Set

$$N^{\uparrow G} = (N^{\uparrow G})e \oplus (\bigoplus_j M_j)$$

where M_j is indecomposable. Let W_j be a vertex of M_j. Then $W_j \leq_G P \cap P^g$ for some $g \in G \backslash N_G(P)$.

Proof. Let $f_H(N)$ be the Green correspondent of N in $N_G(P) \cap H$. Then $f_H(N) = f_H(N)e_P$ by Theorem 5.1. Thus $U := f_H(N)^{\uparrow N_G(P)} = f_H(N)^{\uparrow N_G(P)} e_P$ as well by Lemma 3.2. As any indecomposable direct summand of U has vertex P, we only have to prove that if U is an indecomposable $\theta[N_G(P)]$-module with vertex P such that $U = Ue_P$, then $g(U) = g(U)e$, where $g(U)$ is the Green correspondent of of U in G, as the theorem then follows from Green correspondence. But this follows from (1) applied to $M = g(U)$.

Remark. This generalizes an older result due to Conlon (1964).

Theorem 5.1 also allows us to prove some of the results of Chapter II, Section 7 blockwise:

Lemma 5.8. Let B be a block of $F[G]$ with D as defect group. Then B has only a finite number of isomorphism classes of indecomposable modules if and only if D is cyclic.

Proof: By Corollary II.7.8, as any B-module is D-projective. It also follows from

Corollary 5.9. (Donovan) The maximum complexity of the indecomposable $F[G]$-modules in a block B of $F[G]$ equals the p-rank of a defect group D of B.

Proof: (Alperin and Evens (1981)) As any B-module is D-projective, this follows from Corollary II.7.13 once we prove that some indecomposable B-module M has the property that $c_G(M)$ is equal to the p-rank of D. Let b be the corresponding block of $F[H]$, where $H = N_G(D)$, and let N be a simple b-module. Then N has D as vertex, as D is in the kernel of N, and the Green correspondent M of N in G is in B by Nagao's Theorem 5.1. As $M | N^{\uparrow G}$ and a projective resolution of N induced to G is a projective resolution of $N^{\uparrow G}$, it follows that $c_G(M) \leq c_H(N)$. Similarly, as $N | M_{\downarrow H}$ and projective modules restrict to projective modules, $c_H(N) \geq c_G(M)$, and thus equality holds. The same argument shows that $c_H(N) \geq c_D(N_{\downarrow D})$. However, as $N_{\downarrow D}$ is a trivial $F[D]$-module, this number equals the p-rank of D by a result of Lewis (1968) and we are done by Corollary 7.13.

6. Generalized decomposition numbers.

Definition 6.1. Let x be a p-element of G. The p-section determined by x is the set

(1) $$G(x) = \{g \in G \mid g_p \underset{G}{\sim} x, \quad g_p \text{ the p-part of } g\}$$

In other words, $G(x)$ is the union of all conjugacy classes of G with the property that the p-part of a representative is conjugate to x. Thus we may always choose this representative inside $C_G(x)$.

By a <u>subsection</u> of a block B of $F[G]$, where F is a field of characteristic P, we understand a pair (x,b) where x is a p-element of a defect group D of B and b is a block of $F[C_G(x)]$ such that $Bb \neq 0$. Moreover, if $P \underset{G}{\leq} D$ and b_P is a block of $F[C_G(P)]$ with $b_P Br_P(B) \neq 0$, (P,b_P) is called a B-<u>subpair</u> and b_P is called a <u>root</u> of B. The <u>inertial index</u> of b_P is $|N(b_P) : PC_G(P)|$.

Brauer's Second Main Theorem tells us how to compute $\chi(xy)$ where χ is an irreducible character of G, x is a p-element and $y \in C_G(x)$ is p-regular in terms of the Brauer characters of roots of the block associated with χ and generalized decomposition numbers.

In order to see this, we simply combine Theorem 5.1 and Proposition II.3.6.

In the following, (F,R,S) is a p-modular system such that S is a splitting field of all subgroups of G.

<u>Lemma 6.2</u>. Let $\mathbb{B}$ be a p-block of G, and denote the corresponding block of $F[G]$ by B. Let χ be an irreducible character of $\mathbb{B}$.

Let x be a p-element of G and let $g \in G(x) \cap C_G(x)$. Then

(2) $$\chi(g) = \sum_b \chi_b(g)$$

where b runs through the roots of B in $C_G(x)$ and χ_b is the component of $\chi_{\downarrow C_G(x)}$ in b. In particular, $\chi(g) = 0$ whenever the p-part of g is not conjugate to an element in a defect group of B.

Let $\chi_1,\ldots,\chi_k$ be the irreducible characters of $\mathbb{B}$ and let $x \in D$ where D is a defect group of $\mathbb{B}$ (not necessarily different from 1, although of course we already have complete control over that case). Let $\xi_1,\ldots,\xi_{k_x}$ be the irreducible characters of $C_G(x)$ and set

(3) $$\chi_{i\downarrow C_G(x)} = \sum_r n_{ir}\,\xi_r \,.$$

Now, let $y \in C_G(x)$ be any p-regular element. Then

(4) $$\chi_i(xy) = \sum_r n_{ir}\xi_r(xy) = \sum_r n_{ir}\varepsilon_r\xi_r(y)$$

for some ord(x) 'th root of unity ε_r, as $x \in Z(C_G(x))$.

Next, let $\phi_1,\dots,\phi_{\ell_x}$ be the irreducible Brauer characters of the roots of B in $C_G(x)$ and let $\{d_{rj}\}$ be the decomposition numbers of $C_G(x)$. Thus

(5) $$\xi_r(y) = \sum_j d_{rj}\phi_j(y) \,.$$

Definition 6.3. Same notation as above. By the generalized decomposition number d^x_{ij} we understand the algebraic integer

$$d^x_{ij} = \sum_r n_{ir}\varepsilon_r d_{rj}$$

indexed by the irreducible characters of $\mathbb{B}$ and the irreducible Brauer characters of the roots of B in $C_G(x)$.

We can now formulate (see Brauer (1959))

Brauer's Second Main Theorem 6.4. Let χ_i be an irreducible character of G in the block $\mathbb{B}$. Let x be any p-element, and let $y \in C_G(x)$ be p-regular. Let $\phi_1,\dots,\phi_{\ell_x}$ be the irreducible Brauer characters of $C_G(x)$ and let $\{d^x_{ij}\}$ be the generalized decomposition numbers. Then

(6) $$\chi_i(xy) = \sum_j d^x_{ij}\,\phi_j(y)$$

where $d^x_{ij} \in \mathbf{Z}[\sqrt[m]{1}]$, m = ord(x), and $d^x_{ij} = 0$ whenever ϕ_j does not belong to a root of $\mathbb{B}$ in $C_G(x)$.

Proof: By the analysis above.

This, however, is not the whole story. A lot can be said about the matrix $\underline{\underline{D}}^x = \{d^x_{ij}\}$, as we proceed to demonstrate. Let $\underline{\underline{C}}^x$ denote the Cartan matrix of $C_G(x)$.

For $\underline{\underline{A}}$ a complex matrix, $\overline{\underline{\underline{A}}}$ denotes the complex conjugate and $\underline{\underline{A}}^t$ the transpose.

<u>Theorem 6.5</u>. With the notation above,

$$(\overline{\underline{\underline{D}}}^x)^t \underline{\underline{D}}^x = \underline{\underline{C}}^x \tag{7}$$

while

$$(\overline{\underline{\underline{D}}}^{x_1})^t \underline{\underline{D}}^{x_2} = \underline{\underline{0}} \tag{8}$$

if $x_1 \sim x_2$.

In particular, any column of generalized decomposition numbers is orthogonal to any column of ordinary decomposition numbers.

<u>Proof</u>: Let $y_1, \ldots, y_{\ell_x}$ be representatives of the p-regular conjugacy classes of $C_G(x)$. Set

$$\underline{\underline{X}}^x = (\chi_\alpha(xy_\beta)), \quad \underline{\underline{D}}^x_{\alpha\gamma} = (d^x_{\alpha\gamma}), \quad \underline{\underline{\phi}}^x = (\phi_\gamma(y_\beta)) \tag{9}$$

where χ_α runs through all irreducible characters of G. Then (6) implies

$$\underline{\underline{X}}^x = \underline{\underline{D}}^x \underline{\underline{\phi}}^x \tag{10}$$

since no blocks have roots in common. Now the orthogonality relations yeilds

$$(\overline{\underline{\underline{X}}}^x)^t \underline{\underline{X}} = (\overline{\underline{\underline{\phi}}}^x)^t (\overline{\underline{\underline{D}}}^x)^t \underline{\underline{D}}^x \underline{\underline{\phi}}^x = \begin{Bmatrix} |C_G(xy_1)| & & & 0 \\ 0 & |C_G(xy_2)| & & \\ & & \ddots & \\ 0 & & & |C_G(xy_{\ell_x})| \end{Bmatrix} \tag{11}$$

on the other hand, if $\underline{\underline{\Psi}} = (\xi_i(y_j))$, where $\xi_1, \ldots, \xi_r$ are the irreducible characters of $C_G(x)$, then

$$(12) \qquad \overline{\underline{\underline{\Psi}}}^t \underline{\underline{\Psi}} = (\overline{\underline{\underline{\Phi}}}^x)^t \underline{\underline{C}}^x \underline{\underline{\Phi}}^x = \begin{pmatrix} |C_H(y_1)| & & & 0 \\ 0 & |C_H(y_2)| & & \\ & & \ddots & \\ 0 & & & |C_H(y_{\ell_x})| \end{pmatrix}$$

where $H = C_G(x)$. However, the right hand side of (11) and (12) are identical. Thus the left hand sides are as well. But $\overline{\underline{\underline{\Phi}}}^x$ is invertible as we saw in Theorem I.15.9, and thus (7) follows. Exactly the same argument will show (8).

Notation. The number of irreducible characters in a block $\mathbb{B}$ of G is denoted by $k(\mathbb{B})$, the number of irreducible Brauer characters by $\ell(\mathbb{B})$.

Corollary 6.6. Let $\mathbb{B}$ be a p-block of G. Then

$$(13) \qquad k(\mathbb{B}) - \ell(\mathbb{B}) = \sum_{(x,b)} \ell(b)$$

where the sum is over all proper subsections of $\mathbb{B}$ in G.

Proof: Elementary linear algebra.

Corollary 6.7. Let $\mathbb{B}$ be a p-block of G, and let $x, y \in G$ with non-conjugate p-parts. Then

$$(14) \qquad \sum_{\chi_i \in \mathbb{B}} \chi_i(x)\chi_i(y^{-1}) = 0.$$

Proof: By (8).

Corollary 6.8. (Brauer (1968)) Let $\mathcal{K}$ be a conjugacy class of p-regular elements and $e \in R[G]$ a central idempotent. Then $[\mathcal{K}]e$ is a linear combination of sums of conjugacy classes of p-regular elements.

Proof: Let $\mathcal{K}_1, \ldots, \mathcal{K}_k$ denote the conjugacy classes in G and let $x_i \in \mathcal{K}_i$. Set $\eta_i = \sum_j \chi_j(x_i^{-1})\chi_j$ and recall that $\eta_i(x_j) = \delta_{ij}|C_G(x_i)|$. Now, if $[\mathcal{K}]e = \Sigma\alpha_j[\mathcal{K}_j]$ and x_i is p-singular, then

(15)
$$\alpha_i |G| = \eta_i([\mathcal{K}]e) = \sum_{\substack{j \\ \chi_j(e)\neq 0}} \chi_j(x_i^{-1})\chi_i([\mathcal{K}]) = 0$$

by Corollary 6.7, as we sum over blocks of G

7. Subpairs.

The previous section raises a number of questions one of which is: Given a block B of $F[G]$ and $x \in D$, where D is a defect group of B, what can be said about the roots of B in $C_G(x)$?

It turns out that a particularly nice result holds for the principal block, i.e., the block which contains the trivial 1-dimensional representation.

Notation: We recall that if B is a block of $F[G]$, the corresponding block idempotent is denoted by $\underline{B}$.

Brauer's Third Main Theorem 7.1: Let $B_0 = B_0(G)$ denote the principal p-block of $F[G]$. Then

i) The defect groups of B_0 are the Sylow p-subgroups of G.

ii) The subpairs of B_0 are precisely the subpairs $(P, B_0(C_G(P)))$, where P is any p-subgroup of G.

Proof: (See also Alperin-Broué (1979).) Let $\sigma(G) = \sum_{g \in G} g$. Then for an arbitrary p-subgroup P of G,

(1)
$$\begin{aligned} Br_P(\underline{B}_0)\sigma(C_G(P)) &= Br_P(\underline{B}_0)Br_P(\sigma(G)) \\ &= Br_P(\underline{B}_0\sigma(G)) \\ &= Br_P(\sigma(G)) \\ &= \sigma(C_G(P)) \end{aligned}$$

as $\underline{B}_0$ is the unique block idempotent e for which $e\sigma(G) \neq 0$. In particular, $Br_Q(\underline{B}_0) \neq 0$ for $Q \in Syl_p(G)$, and 1) follows from Corollary 1.11. Moreover, (1) proves that $(P, B_0(C_G(P)))$ is a subpair of B_0 for any p-subgroup P.

Next we prove ii) for $P = Q$. All subpairs (Q,b) of B_0 are conjugate by Brauer's First Main Theorem. But one of them is $(Q,B_0(C_G(Q)))$, which obviously is $N_G(Q)$-invariant.

Finally, let $P < Q$ be arbitrary, and set $R = N_Q(P)$. We may as well assume that $R \in Syl_p(N_G(P))$. Let b be a root of B_0 in $C_G(P)$ and let b' be a block of $C_G(P)R$ such that $Br_P(\underline{b}')\underline{b} = \underline{b}$, which exists by Lemma 3.2 iii). This forces $Br_P(\underline{B}_0)\underline{b}' = \underline{b}'$ as well. We claim that a defect group P' of b' contains P properly. Obviously, $P \leq P'$. Now if $P = P'$, $N_G(P)$ and hence G would have a block B with defect group P by Theorem 3.6 i) and Brauer's First Main Theorem such that moreover $Br_P(\underline{B})\underline{b}' = \underline{b}'$, a contradiction as $Br_P(\underline{B}_0)\underline{b}' = \underline{b}'$, as we have just seen above. Now, $Br_{P'}(\underline{b}') \neq 0$, and as $Br_P(\underline{B}_0)b' = b'$, this yields $\underline{Br}_{P'}(\underline{B}_0)Br_P(\underline{b}') \neq 0$. However, $\underline{B}_0' := Br_{P'}(\underline{B}_0) = \underline{B}_0(C_G(P'))$ by induction as $|Q : P| < |Q : P'|$. Thus $Br_{P'}(\underline{b}') = \underline{B}_0'$, as $\underline{B}_0'$ is primitive in $Z(F[C_G(P')])$. Hence induction furthermore yields $b' = B_0(C_G(P)R)$. But now, the fact that $B_0(C_G(P)P)$ is a root of $B_0(C_G(P)R)$, as we saw above together with the fact that $\underline{b}' = Tr_{N(b)}^{C_G(P)R}(\underline{b})$ see Lemma 3.2) yield that $b = B_0(C_G(P)P)$ as claimed.

<u>Remark</u>. Before we continue, let us briefly recall some standard notation and terminology from group theory. The maximal normal p-subgroup of G is denoted by $O_p(G)$, and the maximal normal p'-subgroup (i.e., of order prime to p) by $O_{p'}(G)$. Set $\bar{G} = G/O_{p'}(G)$. Then G is said to be p-constrained if $C_{\bar{G}}(O_p(\bar{G})) \leq O_p(\bar{G})$.

<u>Corollary 7.2</u>. Let $x \in G$ be a p-element such that $O_{p'}(C_G(x)) = 1$ and $C_G(x)$ is p-constrained. Let B be a p-block of $F[G]$ with D as defect group. Then $x \in_G D$ if and only if $B = B_0(G)$.

<u>Proof</u>: One way is clear by Theorem 7.1 i). Conversely, assume $x \in_G D$. Then $Br_{<x>}(\underline{B}) \neq 0$ by Corollary 1.12. But then $Br_P(\underline{B}) \neq 0$ as well, where $P = O_p(C_G(x))$ by Lemma 2.2. However, $C_G(P)$ is a p-group by assumption and therefore $F[C_G(P)]$ has only one p-block, the principal. Hence $B = B_0(G)$ by Theorem 7.1 ii).

We now introduce

Definition 7.3. A group G is said to be of *deficiency class* d, if

(2) $$d = \max\{d_i \mid |D_i| = p^{d_i}\}$$

where D_i runs through all defect groups of the *non*-principal blocks of G. If this set is empty, G is said to be *deficiency free*.

(Notice, that here we divert from Brauer's terminology (which is $d+1$ rather in all cases), in the hope that it is still possible to change this.)

Corollary 7.2 is very useful when it is combined with the following.

Corollary 7.4. Let G be an arbitrary finite group. Then the following are equivalent:

i) G is deficiency free or of deficiency class 0.

ii) $C_G(x)$ is deficiency free for any element x of order p.

iii) $C_G(x)/\langle x\rangle$ is deficiency free for any element x of order p.

Proof: The equivalence of ii) and iii) are by Lemma 3.10 and 3.11, while the equivalence of i) and ii) follows from Theorem 7.1 ii)

For other results in this direction, see Wales (1970) and Solomon (1974).

Our next goal is to obtain a complete description of the socalled major subsections. For more general results, see Olsson (1982).

Definition 7.5. Let B be a block of $F[G]$ with D as a defect group. Let (u,b_u) be a subsection of B with u D. Then (u,b_u) is called *major* if $u \in Z(D)$.

Lemma 7.6. Let B be a block of $F[G]$ with D as a defect group. Let $P, Q \leq D$ such that

(3) $$PC_D(P) = D = QC_D(Q)$$

and choose subpairs (D,b_D), (P,b_P) and (Q,b_Q) such that

(4) $$b_D Br_D(\underline{b}_P) = b_D = b_D Br_D(\underline{b}_Q).$$

Assume $x \in G$ with $P^x = Q$ and $b_P^x = b_Q$. Then there exists $y \in N(b_D)$ and $z \in C_G(P)$ such that $x = zy$.

<u>Proof</u>: We first observe that (3) and Theorem 3.6 ii) c) assert that D is a defect group of b_P in $PC_G(P)$ and b_Q in $QC_G(Q)$, independent of the choices of b_P and b_Q. Hence, as $b_D Br_D(\underline{B}) = b_D$, it is possible to choose b_P and b_Q to satisfy (4).

Let x be given as above. Then $PC_{D^{x^{-1}}}(P) = D^{x^{-1}}$ and

(5) $$b_{D^{x^{-1}}}(Br_{D^{x^{-1}}}(\underline{b}_P)) = b_{D^{x^{-1}}} Br_{D^{x^{-1}}}(\underline{b}_Q^{x^{-1}}) = b_{D^{x^{-1}}}.$$

Thus $(D^{x^{-1}}, b_{D^{x^{-1}}})$ and (D,b_D) are both subpairs of b_P, so by Brauer's First Main Theorem, they are conjugate in $PC_G(P)$, say by z'. Then $x^{-1}z' \in N_G(D) \cap N(b_D)$, and so does $z'^{-1}x$. Finally, set $z' = zu$, where $z \in C_G(P)$ and $u \in P$. Thus

(6) $$x = z'(z'^{-1}x) = z(uz'^{-1}x)$$

and $y = uz'^{-1}x \in N(b_D)$.

<u>Corollary 7.7</u>. Same notation as above. Let (u,b_u) and (v,b_v) be major subsections with $u, v \in Z(D)$ such that

(7) $$b_D Br_D(\underline{b}_u) = b_D = b_D Br_D(\underline{b}_v).$$

Then (u,b_u) and (v, b_v) are G-conjugate if and only if u and v are $N(b_D)$conjugate.

<u>Proof</u>: One way is trivial by Lemma 7.6, choosing $P = \langle u \rangle$, $Q = \langle v \rangle$. Conversely, if $u^x = v$ where $x \in N(b_D)$, then $b_D Br_D(\underline{b}_u^x) = b_D$.

Thus b_u^x and b_v are both blocks of $F[C_G(v)]$ with the property that D is a defect group and $Br_D(\underline{b}_v)Br_D(\underline{b}_u^x) \neq 0$. Thus $b_v = b_u^x$ by Brauer's First Main Theorem.

In particular, we have proved the following important result on major subsections.

Theorem 7.8. (Brauer (1971) 6A) Let B be a block of F[G] with D as a defect group. Let (D, b_D) be a subpair of B and let $\mathfrak{C}$ be a complete set of representatives of the $N(b_D)$-conjugacy classes of Z(D). For each $u \in \mathfrak{C}$ let b_u be the block of $C_G(u)$ such that (D, b_D) is a subpair of b_u. Then $\{(u, b_u)\}$ forms a complete set of representatives of the major subsections of b in G.

Corollary 7.9. Same notation as above. The number of major subsections of B is at most $|Z(D)|$.

Proof: Clear.

8. Characters in blocks.

We now focus our attention on the characters of an arbitrary block rather than the modules. An important tool is the Cartan matrix.

In the following, we let (F,R,S) be a p-modular system and G an arbitrary finite group. As we intend to examine irreducible characters, we will assume that S is a splitting field of S[H] for all $H \leq G$.

Let $\mathbb{B}$ be an arbitrary p-block of G and B the corresponding block of F[G]. Denote the Cartan matrix of B resp. F[G] by $\underline{\underline{C}}_B$ resp. $\underline{\underline{C}}$ and the decomposition matrix by $\underline{\underline{D}}_B$ resp. $\underline{\underline{D}}$.

Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G and $\phi_1, \ldots, \phi_\ell$ the irreducible Brauer characters, and choose notation so that the first k_B resp. ℓ_B of them are those of $\mathbb{B}$. Finally, let $x_1, \ldots, x_\ell$ be representatives of the p-regular conjugacy classes in G, and set $\underline{\Phi} = \{\Phi_i(x_j)\}$, $\underline{\hat{\Phi}} = \{\phi_i(x_j)\}$.

Theorem 8.1. (Brauer (1941)) The determinant of $\underline{\underline{C}}$ is a power of p.

Proof: (Alperin, Collins and Sibley (1983)) We have already seen in Theorem I.15.9 that $\det(\underline{\underline{C}}) \neq 0$. Moreover, $\det(\underline{\underline{C}}) \in \mathbf{Z}$ by definition and is positive by Theorem I.15.5. If $|G|$ is prime to p, then $\underline{\underline{C}}$ is the identity matrix. Assume therefore that p divides $|G|$. The proof is divided into two separate cases:

a) $O_p(G) \neq 1$. Set $H = O_p(G)$. Let $\{\Phi_i\}$ resp. $\{\overline{\Phi}_i\}$ denote the characters of the p.i.m. 's of $F[G]$ resp. $F[\overline{G}]$, where $\overline{G} = G/H$. (Observe that $F[G]$ and $F[\overline{G}]$ have the same number of isomorphism classes of simple modules. Now if ϕ_H denotes the Brauer character of the $F[G]$-module $F[H]$, then $\Phi_i = \overline{\Phi}_i \phi_H$ by Theorem II.11.14. Moreover, $\phi_H(x) = |C_H(x)|$ for all p-regular elements $x \in G$. Let $\overline{\underline{\underline{C}}}$ denote the Cartan matrix of $F[\overline{G}]$ and set $\overline{x}_i = x_i H$ with the notation above. Then $\overline{x}_1, \ldots, \overline{x}_\ell$ are representatives of the p-regular conjugacy classes in $\overline{G}$. Finally, set $\overline{\underline{\underline{\Phi}}} = \{\overline{\Phi}_i(\overline{x}_j)\}$. Then

$$\underline{\underline{\Phi}} = \begin{pmatrix} |C_H(x_1)| & & & \\ & |C_H(x_2)| & & \\ & & \ddots & \\ & & & |C_H(x_\ell)| \end{pmatrix} \overline{\underline{\underline{\Phi}}}$$

and as $\underline{\underline{\Phi}} = \underline{\underline{C}}\,\underline{\underline{\phi}}$ while $\overline{\underline{\underline{\Phi}}} = \overline{\underline{\underline{C}}}\,\overline{\underline{\underline{\phi}}}$, the fact that $\det(\underline{\underline{\Phi}}) \neq 0$ implies that

$$\det(\underline{\underline{C}}) = \prod_{i=1}^{\ell} |C_H(x_i)| \det(\overline{\underline{\underline{C}}}) \tag{2}$$

and thus we are done by induction.

b) $O_p(G) = 1$. The fact we want to prove is equivalent to the following: Let M be any $F[G]$-module and let ψ be the Brauer character of M. Then $p^n \psi \in \mathrm{Span}_{\mathbf{Z}}\{\Phi_i\}$ for some $n \in \mathbf{N}$ as a class function on the p-regular elements of G. By Lemma II.4.3, it suffices to prove this for a module of the form $K\uparrow_N^G$ where $N < G$ and K is an $F[N]$-module. Let ψ_K be the Brauer character of K. By induction, there exists an m such that $p^m \psi_K \in \mathrm{Span}_{\mathbf{Z}}\{\Psi_j\}$, where $\{\Psi_j\}$ are the characters of the p.i.m. 's of $F[N]$. Thus $p^m \psi_K \uparrow_N^G \in \mathrm{Span}_{\mathbf{Z}}\{\Phi_i\}$ and we are done.

Remark. The proof does not give any information about the magnitude of $\det(\underline{\underline{C}})$. However, this can be obtained by elementary methods. First we observe

Lemma 8.2. The determinant of $\underline{\underline{\Phi}}$ is a unit in R.

Proof: This is just a variation of the proof of Theorem I.15.9. Let $Q_i \in \mathrm{Syl}_p(C_G(x_i))$ and let i be arbitrary but fixed. Let $\zeta_1,\ldots,\zeta_r$ denote the irreducible characters of $\langle x_i\rangle$ inflated to $\langle x_i\rangle \times Q_i$ and set $\xi_i = \sum_t \zeta_t(x_i^{-1})\zeta_t$. It is easily checked that

(3) $$\xi_i^{\uparrow G}(x_j) = \delta_{ij}\,\frac{|C_G(x_i)|}{|Q_i|} .$$

This implies that $\{\chi_i\}$ considered as class functions on the p-regular elements span a free module over R of rank ℓ, which reduced modulo (π) has dimension ℓ. Thus the same holds for the irreducible Brauer characters, and the lemma follows, as does

Corollary 8.3. Let $\bar{\phi}_i(x_j) = \phi_i(x) + (\pi)$. Thus $\{\bar{\phi}_i\}$ are the traces of the irreducible representations of G over F. Then

i) The $\bar{\phi}_i$'s are linearly independent over F.

ii) The elementary divisors of the decomposition matrix of G are prime to p.

By means of Theorem 8.1, we may now improve Corollary 8.3 ii) to

Corollary 8.4. The elementary divisors of the decomposition matrix of G are all 1.

Proof: We have seen that the rank of the decomposition matrix $\underline{\underline{D}}$ is ℓ. Let q be a prime different from p. Then the q-rank of $\underline{\underline{D}}$ is ℓ as well by Theorem 8.1, as $\underline{\underline{D}}^t\underline{\underline{D}} = \underline{\underline{C}}$. Corollary 8.4 now follows from Corollary 8.3.

Corollary 8.6. As class functions on the set of p-regular elements of G,

(4) $$\mathrm{Span}_{\mathbf{Z}}(\{\chi_i\}_{\mathbb{B}}) = \mathrm{Span}_{\mathbf{Z}}(\{\phi_j\}_{\mathbb{B}}).$$

Proof: Clear.

We may now prove

Corollary 8.7. The elementary divisors of $\underline{C}'$ are the numbers $|C_G(x_1)|_p, \ldots, |C_G(x_\ell)|_p$. In particular, $|G|_p \underline{C}^{-1} \in \mathrm{Mat}_\ell(\mathbf{Z})$.

Proof: Recall from Theorem I.15.9 that

$$(5) \qquad \bar{\underline{\Phi}}^t \underline{C} \underline{\Phi} = \begin{pmatrix} |C_G(x_1)| & & & \\ & |C_G(x_2)| & & \\ & & \ddots & \\ & & & |C_G(x_\ell)| \end{pmatrix}$$

However, as $\underline{\Phi}$ is invertible in $\mathrm{Mat}_\ell(R)$ by Lemma 8.2, the elementary divisors of $\underline{C}$ over R are the claimed numbers. Hence the same is true over $\mathbf{Z}$ by Theorem 8.1.

We are now ready to concentrate on the characters. Let B be of defect d. Set $|G| = p^a n$, where $(p,n) = 1$ and recall that as $\mathbb{B}$ is of defect d, p^{a-d} divides $\chi_i(1)$ for all $i=1,\ldots,k_B$. We recall that the height h_i of χ_i is defined by $\chi_i(1) = p^{a-d+h_i} n_i$ where $(p,n_i) = 1$.

Let χ be an arbitrary irreducible character of $\mathbb{B}$ and assume $\mathbb{B}$ is of defect d. Define the class function χ^* by

$$(6) \qquad \chi^*(x) = \begin{cases} p^d \chi(x) & \text{for } x \text{ p-regular} \\ 0 & \text{otherwise} \end{cases}$$

The following is well-known.

Theorem 8.8. Let h denote the height of χ. Then $\frac{1}{p^h}\chi^*$ is a generalized character, while $\frac{1}{p^{h+1}}\chi^*$ is not.

Notation. Recall from Chapter I, Section 15, that the set of p-regular elements in G is denoted by G_0, and that if η, ζ are class functions of G into S, we set

(7) $$(\eta,\zeta)_{G_0} = \frac{1}{|G|} \sum_{x \in G_0} \eta(x)\zeta(x^{-1}).$$

Proof of Theorem 8.8: We first prove that $p^{a-d}\chi^*$ is a generalized character. Let $\chi^* = \Sigma a_i \chi_i$ where $a_i \in S$. Then

(8) $$(p^{a-d}\chi^*, \chi_i)_G = p^a(\chi,\chi_i)_{G_0}$$

for all i. Thus it suffices to prove that for all irreducible Brauer characters ϕ, ψ of G, $p^a(\phi,\psi)_{G_0} \in \mathbf{Z}$. But this follows from Corollary 8.7.

To improve this, we just compute

(9) $$\left(\frac{1}{p^h}\chi^*, \chi_i\right)_G = \frac{p^{d-h}}{|G|} \sum_{x_j} |C_G(x_j)|\chi(x_j)\chi_i(x_j^{-1})$$
$$= \frac{p^{d-h}}{|G|} \sum_j \chi(1)\omega([\mathcal{K}_j])\chi_i(x_j^{-1})$$

where ω is the central character corresponding to χ, $\mathcal{K}_j$ is the conjugacy class containing x_j and $[\mathcal{K}_j] = \sum_{x \in \mathcal{K}_j} x$. Let $\chi(1) = p^{a-d+h}m$, where $(p,m) = 1$. Then

(10) $$\left(\frac{1}{p^h}\chi^*, \chi_i\right)_G = \frac{m}{n} \sum_j \omega([\mathcal{K}_j])\chi_i(x_j^{-1}) .$$

Thus $n(\frac{1}{p^h}\chi^*, \chi_i)$ is an algebraic integer. As $p^{a-d}(\chi^*,\chi_i) \in \mathbf{Z}$ as well, it follows in fact that $(\frac{1}{p^h}\chi^*,\chi_i) \in \mathbf{Z}$. Finally, that $\frac{1}{p^{h+1}}\chi^*$ is not a generalized character follows from the fact that the degree of a generalized character which vanishes on p-singular elements must be divisible by the order of a Sylow p-subgroup.

Remark. Observe Theorem 8.8 was obtained without appealing to Brauer's characterization of characters, thanks to the proof of Theorem 8.1.

Corollary 8.9. Let G be a finite group and χ an irreducible character of G. Let the p-part of $|G|$ be p^a, that of χ p^b. Define the class function $\tilde{\chi}$ by

$$(11) \qquad \tilde{\chi}(x) = \begin{cases} p^{a-b}\chi(x) & \text{for } x \text{ p-regular} \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{\chi}$ is a generalized character.

Proof: Clear.

Remark: Notice that the statement does not require any concepts from modular representation theory. Notice also that this is a generalization of Corollary I.16.2 and that no proof based on ordinary character theory is known.

The construction of χ^* from χ in (6), which is due to Brauer (1953) is very important and will lead to a number of interesting observations.

Notation. Set

$$(12) \qquad a_{ij} = (\chi_i^*, \chi_j)_G = (\chi_i, \chi_j^*)_G = p^d(\chi_i, \lambda_j)_{G_0}$$

for $i, j \leq k_B$, and set $\underline{\underline{A}} = \{a_{ij}\}$.

Lemma 8.10. The matrix $\underline{\underline{A}}$ has the following properties

i) $\underline{\underline{A}}$ is integral and symmetric

ii) $\underline{\underline{A}} = p^d \underline{\underline{D}}_B \underline{\underline{C}}_B^{-1} \underline{\underline{D}}_B^t$

iii) $\underline{\underline{A}}^2 = p^d \underline{\underline{A}}$

iv) $Tr(A) = p^d \ell_B$

Proof: i) and ii) are by definition,
iii) follows from ii),
iv) follows from ii), as

$$(13) \qquad Tr(\underline{\underline{A}}) = p^d Tr(\underline{\underline{D}}_B \underline{\underline{C}}_B^{-1} \underline{\underline{D}}_B^t) = p^d Tr(\underline{\underline{D}}_B^t \underline{\underline{D}}_B \underline{\underline{C}}_B^{-1}).$$

Another way of stating iii) is

Lemma 8.11. Let χ', χ'' be irreducible characters in different blocks. Then $(\chi',\chi'')_{G_0} = 0$. In particular,

$$\chi_i^* = \sum_{j=1}^{k_B} a_{ij}\chi_j \tag{14}$$

and

$$p^d a_{ij} = \sum_s a_{is}a_{sj} \; . \tag{15}$$

Proof: The first observation follows from the fact that if ϕ', ϕ'' are irreducible Brauer characters in different blocks, then $(\phi', \phi'')_{G_0} = 0$ as we saw in Theorem I.15.9, while (15) is just iii) above.

Notation. Denote the number of irreducible characters in $\mathbb{B}$ of height i by k_i. Recall that $k_0 \neq 0$ and let χ_{i_0} be one of height 0. The central character associated with χ_i is denoted by ω_i.

The following goes back to Brauer (1953) and Brauer and Feit (1959) and is partly inspired by Hansen (1983).

Theorem 8.12. With the notation above,

i) $a_{ii} \neq 0$ and $a_{ii} \not\equiv 0 \bmod p^{h_i+1}$ if and only if χ_i is of height 0. Moreover p^{h_i} divides a_{ij} for all j. In particular, $a_{ij}\dfrac{|G|}{p^d\chi_i(1)} \in \mathbf{Z}$.

ii) For all i, j, s, we have

$$a_{ij}\frac{|G|}{p^d\chi_i(1)} \equiv a_{sj}\frac{|G|}{p^d\chi_s(1)} \bmod p. \tag{16}$$

iii) For all j we have

$$a_{i_oj}\chi_{i_o}(1) \equiv a_{i_oi_o}\,\chi_j(1) \bmod p^{a-d+h_j+1} \tag{17}$$

In particular, $a_{i_oj} \neq 0$ for all j, and the p-part of a_{i_oj} is precisely p^{h_j}.

iv) The rank of $\underline{\underline{A}}$ reduced modulo p is 1.

Proof: That $a_{ii} \neq 0$ follows from the fact that $(\chi_i^*, \chi_i)_G = p^d a_{ii}$. Moreover,

(18)
$$a_{ij} = \frac{p^d}{|G|} \sum_{x \in G_0} \chi_i(x)\chi_j(x^{-1})$$
$$= \frac{p^d}{|G|} \sum_t |C_G(x_t)|\chi_i(x_t)\chi_j(x_t^{-1})$$
$$= \frac{p^d}{|G|} \chi_i(1) \sum_t \omega_i([\mathcal{K}_t])\chi_j(x_t^{-1})$$

which shows that p^{h_i} divides a_{ij}, as $\omega_i([\mathcal{K}_t])$ is an algebraic integer. Thus $a_{ij} \frac{|G|}{p^d \chi_i(1)} \in \mathbf{Z}$, and

(19)
$$a_{ij} \frac{|G|}{p^d \chi_i(1)} \equiv a_{sj} \frac{|G|}{p^d \chi_s(1)} \bmod p$$

for all i, j, s by (18), as $\omega_i([\mathcal{K}_t]) \equiv \omega_s([\mathcal{K}_t]) \bmod p$ by Theorem 1.13.6, which proves ii). Choosing $i = j = i_o$ we moreover see that if $a_{i_o i_o} \equiv 0 \bmod p$, then $a_{s i_o} \equiv 0 \bmod p$ for all s and then $\frac{1}{p} \chi_{i_o}^*$ would be a generalized character, a contradiction by Theorem 8.8. Finally, if we choose $j = i$ and $s = i_o$ in (19), we see that if $h_i \neq 0$, then $a_{ii} \frac{|G|}{p^d \chi_i(1)} \equiv 0 \bmod p$, and all of i) is proved.

iii) is an immediate consequence of ii).

iv) Again ii) implies that

(20)
$$a_{sj} \equiv a_{i_o j} \frac{\chi_s(1)}{\chi_{i_o}(1)} \bmod p^{h_s+1} R$$

as $\frac{\chi_s(1)}{\chi_{i_o}(1)} \in R$. Thus the s'th row of $\underline{\underline{A}}$ is a multiple of the i_o'th mod pR, and iv) follows.

We may now determine the elementary divisors of $\underline{\underline{C}}_B$.

Corollary 8.13. Exactly one of the elementary divisors of $\underline{\underline{C}}_B$ is p^d, while the others are strictly smaller.

Proof: We have seen that $(\chi_i^*,\chi_j)_G = p^d(\chi_i,\chi_j)_{G_0}$ is an integer for all $i, j \leq k_B$. Hence $p^d(\phi_i,\phi_j)$ is an integer for all $i, j \leq \ell_B$ by Corollary 8.6 and consequently $p^d \underline{\underline{C}}_B'^{-1} \in \mathrm{Mat}_{\ell_B}(\mathbf{Z})$, which is equivalent to the fact that all elementary divisors of $\underline{\underline{C}}_B$ divides p^d. However, as the rank of $\underline{\underline{A}}$ reduced modulo p is 1, the same obviously holds for $p^d \underline{\underline{C}}_B^{-1}$ and thus only one elementary divisor is actually 1.

Theorem 8.12 has a number of other applications, such as

Corollary 8.14. (Brauer and Feit (1959))

i) $k_B \leq \frac{1}{4}(p^{2d}-1) + 1$ for p odd.

ii) $k_B \leq 2^{2(d-1)}$ for $p = 2$ and $d > 1$.

Proof: By definition,

$$(\chi_i^*,\chi_i^*) = p^d a_{ii} = a_{ii}^2 + \sum_{j\neq 1} a_{ij}^2. \tag{21}$$

Moreover, if we choose $i = i_o$, we know that $a_{i_o j} \neq 0$ for all j and thus

$$k_B \leq \sum_{j\neq i_o} a_{i_o j}^2 + 1 = p^d a_{i_o i_o} - a_{i_o i_o}^2 + 1 . \tag{22}$$

The right hand side assumes its maximal value for $a_{i_o i_o} = \frac{p^d}{2}$. As $a_{i_o i_o} \in \mathbb{N}$ and p does not divide $a_{r_o r_o}$, the statement follows.

Corollary 8.15. (Brauer (1941), Brauer and Feit (1959)) If the height of χ_i is h_i and $\mathbb{B}$ is of defect d, then

i) $h_i \leq d - 2$ if $d \geq 2$

ii) $h_i = 0$ if $d \leq 2$.

Proof: By Theorem 8.12 i) and Proposition I.16.1, $h_i+1 < d$ if $h_i \neq 0$.

Corollary 8.16. (Landrock (1981a) Assume $p = 2$. Then

i) Assume $d > 1$. Then $k_0 \equiv 0 \bmod 4$.

ii) Assume $d > 2$. Suppose furthermore $k_{d-2} \neq 0$. Then

a) $k_0 = 4$

b) $k_{d-2} \leq 3$

Proof: i) follows from (21), as $a_{ij}^2 \equiv 0$ or $1 \bmod 4$, always

ii) Let χ_r be of height $d-2$. By i) there exists at least four irreducible characters $\chi_1,\dots,\chi_4$ in $\mathbb{B}$ of height 0. By Proposition 8.12 i) and iii), 2^{d-2} divides a_{ri}, $i = 1,\dots,4$, while 2^{d-1} divides a_{rr}. Thus

$$(23) \qquad 2^{2(d-1)} \leq \sum_{i=1}^{4} a_{ri}^2 + \sum_{\substack{i>4 \\ i\neq r}} a_{ri}^2 = 2^d a_{rr} - a_{rr}^2 \leq 2^{2(d-1)} .$$

Hence equality holds and $\sum_{\substack{i>4 \\ i\neq r}} a_{ri}^2 = 0$. In particular, $k_0 = 4$ by Theorem 8.12 iii). Finally, if we set $i = 1$, say, in (21), we see that $k_{d-2} \leq 3$.

Remark: For other applications of Lemma 8.10, 11 and Proposition 8.12, see Olsson (1981). See Broué (1980) for other results.

Corollary 8.14 through 16 are all based on properties of characters on p-regular elements. It seems reasonable also to try to use their values on p-singular elements, in particular since we have Brauer's Second Main Theorem at our disposition.

Following Brauer (1968), we therefore generalize the idea behind the matrix $\underline{\underline{A}}$.

We continue with the previous notation. Moreover, we let $w \in D$ (including 1 as a possibility) be fixed and set $C = C_G(w)$. Let $\{b_r\}$ be the roots of $\mathbb{B}$ in $F[C]$ and denote the component of $\chi_{i\downarrow C}$ in b_r by $\chi_i^{b_r}$. In other words, $\chi_i^{b_r}(x) = \chi_i(\underline{b}_r x)$ for all $x \in C$, where as earlier the block idempotent of a block B is denoted by $\underline{B}$. Recall that if g belongs to the p-section $G(w)$ then $g \underset{G}{\sim} wx$ for some p-regular element $x \in C$. Moreover, by Lemma 6.2,

(24) $$\chi_i(g) = \sum_r \chi_i(\underline{b}_r\, wx) = \sum_r \chi_i^{b_r}(wx).$$

We now set

(25) $$m_{ij}(w)^{b_r} = m_{ij}^{b_r} = \frac{1}{|C|} \sum_{x \in C_0} \chi_i^{b_r}(wx)\chi_j^{b_r}(w^{-1}x^{-1})$$

and $\underline{\underline{M}}(w)^{b_r} = \underline{\underline{M}}^{b_r} = \{m_{ij}^{b_r}\}$. It follows that if $1=w_1,\dots,w_q$ are representatives of the G-conjugacy classes in D, then

(26) $$\sum_{w_s} \sum_{r_s} m_{ij}(w)^{b_{r_s}} = (\chi_i,\chi_j)_G = \delta_{ij} .$$

The numbers $m_{ii}(w)^{b_r}$ are called the <u>contributions</u> of χ_i. We now have, just as in Lemma 8.10,

<u>Lemma 8.17</u>. (Brauer (1968), (5C)) The matrices $\underline{\underline{M}}(w)^{b_r}$ satisfies

i) $\underline{\underline{M}}(w)^{b_r} = \underline{\underline{D}}^w_{b_r}\, \underline{\underline{C}}^{-1}_{b_r} (\overline{\underline{\underline{D}}}^w_{b_r})^t$

ii) $(\underline{\underline{M}}(w)^{b_r})^2 = \underline{\underline{M}}(w)^{b_r}$

iii) $\mathrm{Tr}(\underline{\underline{M}}(w)^{b_r}) = \ell(b_r)$

iv) $\underline{\underline{M}}(w)^{b_r}$ is symmetric and the entries of $p^{d(b_r)}\underline{\underline{M}}(w)^{b_r}$ are algebraic integers

v) $p^d \underline{\underline{M}}(1)^B = \underline{\underline{A}}$.

<u>Proof</u>: By Brauer's Second Main Theorem.

We proceed to investigate these contributions.

<u>Notation</u>. Following Brauer (1968) (p.903), we set

(27) $$\omega_i^{b_r}([\mathcal{K}_g]) = \begin{cases} \dfrac{|G|}{|C|}\dfrac{\chi_i^{b_r}(wx)}{\chi_i(1)} & \text{if } g \underset{G}{\sim} wx,\ x \in C_0 \\ 0 & \text{otherwise} \end{cases}$$

Denote the irreducible characters in the roots of $\mathbb{B}$ in C by $\{\tilde{\chi}_j\}$. Thus

(28) $$\chi_i(wx) = \Sigma c_{ij}\varepsilon_j\tilde{\chi}_j(x)$$

for $x \in C_0$, ε_j an ord(w)'th root of unity and $c_{ij} = (\chi_i, \tilde{\chi}_j)_C$ by (24). Hence,

(29) $$\omega_i([\mathcal{K}_{wx}]) = \frac{|G|}{|C|} \sum_j \frac{\tilde{\chi}_j(1)}{\chi_i(1)} c_{ij}\varepsilon_j\tilde{\omega}_j([\tilde{\mathcal{K}}_x])$$

where $\mathcal{K}_g$ resp. $\tilde{\mathcal{K}}_g$ is the G- resp. C-conjugacy conjugacy class containing $g \in C$. Let u_{ij} denote the coefficient to $\tilde{\omega}_j$ in (29).

<u>Remark</u>: Note that $u_{ij} \in R$ if χ_i is of height 0 and that if this is the case then furthermore $u_{ij} \equiv 0 \bmod p^{d-d(b_r)}$ if $d(b_r) < d$, for $\tilde{\chi}_j$ in the block determined by b_r.

<u>Lemma 8.18</u>. (Brauer (1968), (3D)). With the notation above, we have

i) $\omega_i^{b_r}([\mathcal{K}_{wx}]) \in R$ for all $x \in C_0$

ii) $\omega_i^{b_r}([\mathcal{K}_{wx}]) \equiv \omega_j^{b_r}([\mathcal{K}_{wx}]) \bmod (\pi)$ for all $x \in C_0$.

<u>Proof</u>: For $x \in C_0$, let $\tilde{\mathcal{K}}_x$ denote the C_0-conjugacy class containing x. Set $f_i([\tilde{\mathcal{K}}_x]) = \omega_i([\mathcal{K}_{wx}])$ for all $x \in C_0$. Then

(30) $$f_i([\tilde{\mathcal{K}}_x]) = \sum_r \omega_i^{b_r}([\dot{\mathcal{K}}_{wx}])$$

$$= \sum_j u_{ij}\tilde{\omega}_j([\tilde{\mathcal{K}}_x])$$

where u_{ij} is as above (see (29)). Thus f_i is S-linear, and it follows that

(31) $$f_i([\tilde{\mathcal{K}}_x]\underline{b}_r) = \omega_i^{b_r}([\mathcal{K}_{wx}]).$$

But $[\tilde{\mathcal{K}}_x]\underline{b}_r$ is an R-linear combination of class sums of p-regular elements in C by Corollary 6.8, and as $f_i([\tilde{\mathcal{K}}_x]) \in R$ for all $x \in C_0$ by definition, it follows that i) holds.

To see ii), recall that

(32) $$\pi^{-1}(f_{i_1}[\tilde{\mathcal{K}}_x] - f_{i_2}[\tilde{\mathcal{K}}_x]) \in R$$

for all $i_1, i_2 \leq k_B$. But then the argument above yields that

(33) $$\pi^{-1}(\omega_{i_1}^{b_r}(wx) - \omega_{i_2}^{b_r}(wx)) \in R$$

which is equivalent to ii).

Remark: It follows from our remark just prior to Lemma 8.18 that if χ_i is of height 0, then actually $\omega_i^{b_r}([\mathcal{K}_{wx}]) \equiv 0 \bmod (p^{d-d(b_r)})$, so this lemma is only useful when applied to major subsections, which is exactly what we proceed to do.

Theorem 8.19. (Brauer (1968), 4B) Let (w,b_r) be a major subsection, and let b be a block of $F[C_G(D)]$ such that $Br_D(\underline{b}_r)b = b$. Then

(34) $$\omega_i^{b_r}(w) \equiv |N(b) : N(b) \cap C| \bmod p$$

for all i. In particular, $\chi_i^{b_r}(w) \neq 0$ and consequently $m_{ii}(w)^{b_r} \neq 0$, for all i.

Proof: By Lemma 8.18, we may as well assume that χ_i is of height 0. Let the order of a Sylow p-subgroup of C be p^c. Let $\tilde{B}$ be the block of $F[N]$, where $N = N_G(D)$, corresponding to B. As an R-form of χ_i has D as vertex, Corollary 5.2 and Green Correspondence yield

(35) $$\chi_i(1) \equiv \frac{|G|}{|N|}\chi_i(\underline{\tilde{B}}) \bmod p^{a-d+1}$$

$$= \frac{|G|}{|N|}\chi_i(Tr_{N(b)}^{N}(b)) \bmod p^{a-d+1}$$

by Lemma 3.2

$$= \frac{|G|}{|N(b)|}\chi_i(\underline{b}) \bmod p^{a-d+1} .$$

Likewise

$$(36) \qquad \chi_i(b_r) \equiv \frac{|C|}{|C \cap N|} \chi_i(\mathrm{Tr}_{C \cap N(b)}^{C \cap N}(\underline{b})) \mod p^{c-d+1}$$

$$= \frac{|C|}{|C \cap N(b)|} \chi_i(\underline{b}) \mod p^{c-d+1} \quad .$$

Now (35) and (36) yield (34)

We may now prove Brauer's Theorem on blocks with abelian defect groups and inertial index 1. This is based on the following observation.

Lemma 8.20. Suppose $\ell_B = 1$. Then $k_B \leq p^d$ and the Cartan matrix is $\{p^d\}$.

Proof: First of all $C_B = \{p^d\}$ by Corollary 8.13. Let $\{d_i\}$, $i = 1,\ldots,k_B$ be the decomposition matrix. As $\Sigma d_i^2 = p^d$ it follows that $k_B \leq p^d$.

Theorem 8.21. (Brauer (1971), Prop. (6G)) Assume the defect group D of B is abelian and that the inertial index of B is 1. Then $k_B = p^d$, $\ell_B = 1$, $C_B = \{p^d\}$ and the decomposition numbers are all 1. Moreover the number of major subsections is p^d.

Proof: (See Hansen (1983)) By Corollary 7.10 and Theorem 7.9, B has exactly p^d major subsections. Hence

$$(37) \qquad p^d \geq \sum_{(w,b_r)} p^d m_{ii}^{b_r}(w) \geq p^d \Big(\prod_{(w,b_r)} |p^d m_{ii}^{b_r}(w)| \Big)^{\frac{1}{p^d}} \geq p^d$$

and thus

$$(38) \qquad \prod_{(w,b_r)} (p^d m_{ii}^{b_r}(w)) = 1 \ .$$

As $p^d m_{ii}^B(1)$ is an integer, this forces $p^d m_{ii}^B(1) = 1$. As this holds for all i's, Lemma 8.10 iv) yields that

$$(39) \qquad \ell_B p^d = k_B = p^d$$

where the last equality follows from (22) since $a_{ii} = 1$ for all i.

Corollary 8.22. (Landrock (1981a), Lemma 1.7) Assume the defect group D of B is abelian and let b be a root of B in $DC_G(D)$. Assume furthermore that $C_G(w) \cap N(b) = C_G(D)$ for some $w \in D$.

Let b_1 be a root of B in $C_G(w)$. Then $\ell(b_1) = 1$. Denote the corresponding generalized decomposition numbers by $\{d_i^w\}$. Then $d_i^w \neq 0$ for all i, and $\sum_i |d_i^w|^2 = |D|$.

In particular, $k_B \leq |D|$.

Proof: Our assumption conveniently implies that the inertial index of b_1 is 1. Thus $\ell(b_1) = 1$ by Theorem 8.22, and $\sum_i |d_i^w|^2 = D$ by Brauer's Second Main Theorem. Moreover, $d_i^w \neq 0$ by Theorem 8.19. Set $a = \prod_i |d_i^w|^2$ and let $\{a^{\rho_i}\}$, $i=1,\ldots,v$ be the algebraic conjugate of a over $\mathbf{Z}$. Then

$$(40) \qquad v|D| \geq vk_B |\prod_i a^{\rho_i}|^{\frac{1}{vk_B}} \geq vk_B$$

as $\prod a^{\rho_i} \in \mathbf{Z}\setminus\{0\}$, and the last statement follows.

Remark: The assumption above on $C_G(w) \cap N(b)$ of course may be replaced by $\ell(b_1) = 1$, in which case we do not even need D to be abelian provided (w,b_1) is a major subsection. But if D is non-abelian, the assumption that $C_G(w) \cap N(b) = DC_G(D)$ will not necessarily imply that $\ell(b_1) = 1$.

Example 1. Let $G = GL(2,3)$, let F be of characteristic 2 and let w denote the central involution of G, D a Sylow 2-subgroup. Then the principal 2-block B_0 of G has inertial index 1 and $N(B_0(C_G(D)) = DC_G(D) = D$. However, $\ell_{B_0} = 2$.

Remark. We have just seen that if a block B has inertial index 1, then we cannot always deduce $\ell_B = 1$. The converse conclusion cannot be drawn either, in general, not even for p-solvable groups.

Our remarks above indicate that to generalize Brauer's Theorem 8.22 to the non-abelian case might be complicated. However, one step towards this has been taken in Broué and Puig (1980). The class of blocks they describe are what they call nilpotent blocks. Here the p-block B with defect group D is called nilpotent if for all $P \leq D$ and all roots b of B in $C_G(P)$ we have that $N(b)/C_G(P)$ is a p-group. An important step towards this description is the following interesting result (compare Theorem 8.19).

Theorem 8.23. (Broué and Piug (1980), Theorem 1.5) Let $\mathbb{B}$ be an arbitrary p-block of G with B the corresponding block of $F[G]$, and let (w,b_r) be a major subsection. Then $p^{d(b_r)} m_{ii}(w)^{b_r}$ is a unit in R, with the previous notation, for all i. In particular, all contributions from major subsections are non-vanishing.

It does not seem to be possible to prove this directly from the methods which yielded Theorem 8.19. The result is based on some relatively deep results of Broué (1978) and we refer the reader to the original sources for a proof.

We end this section with a description of the algebra structure of the block considered in Theorem 8.22, which explains the somewhat surprising (at that stage) result in Corollary II.16.1, hopefully to our complete satisfaction.

The results below have been proved in general for nilpotent blocks in Broué and Puig (1980), while the specific results as we state them have been obtained independently by Külshammer (1980). We shall follow his work quite closely and start with

Lemma 8.24. Let B be a block of $F[G]$ with D as defect group and assume B has exactly one simple module. Denote its dimension by m. Let B_0 be the basic algebra of B. Then $\dim_F B_0 = |D|$, and $B \simeq \mathrm{Mat}_m(B_0)$.

Proof: Let $P(B)$ denote the p.i.m. of B. Then $B_0 \simeq (P(B),P(B))^B$ by definition, and as $B_B \simeq P(B)^{(m)}$, we see that $B \simeq \mathrm{Mat}_m(B_0)$. Moreover, $\dim_F B_0$ is the Cartan invariant of B, which is $|D|$ by Corollary 8.13.

Lemma 8.25. Let B be a block of $F[G]$ with defect group D, and assume $G = DC_G(D)$. Then $\ell(B) = 1$. Moreover, with the notation of Lemma 8.24, $B_0 \simeq F[D]$. In particular, $B \simeq \mathrm{Mat}_m(F[D])$.

Proof: As any simple B-module has D in the kernel and is a projective $F[G/D]$-module, $\ell(B) = 1$ by Lemma 3.9. The corresponding block $b = \underline{B}F[C_G(D)]$ is simple and isomorphic to $\mathrm{Mat}_m(F)$. Let $\underline{B} = \Sigma e_i$ be a primitive idempotent decomposition of $\underline{B}$ in $F[C_G(D)]$. As $B/J(B) \simeq b$, this is also a primitive idempotent decomposition in $F[DC_G(D)]$. Thus $B_0 \simeq e_1F[G]e_1$. Hence $\Gamma : F[D] \to B_0$ defined by $\Gamma(a) = e_1ae_1 = ae_1$ is an algebra homomorphism, and injective as $e_1B_{\downarrow D}$ is is free. As $F[D]$ and B_0 have the same dimension, it is an isomorphism.

Theorem 8.26. Same assumption as in Theorem 8.21, and notation as in Lemma 8.24. Then $B \simeq \mathrm{Mat}_m(F[D])$.

Proof: As we saw in Lemma 8.24, this is equivalent to proving that $B_0 \simeq F[D]$. Now, to see this, we first observe that $Z(B_0) \simeq Z(B)$ which by Theorem 8.21 is of dim D'. Thus B_0 is abelian and naturally isomorphic to $Z(B)$ by Lemma 8.24.

Set $C = C_G(D)$ and $N = N_G(D)$, and let b be a root of B in $F[C]$. Then $b \simeq \mathrm{Mat}_{m(b)}(F[D])$, where $m(b)$ is the dimension of the simple b-module E, by Lemma 8.25.

Let $\tilde{B}$ denote the corresponding block of $F[N]$. Then $E^{\uparrow N}$ is the simple $\tilde{B}$-module, as C is the inertial group of M by assumption. Hence $\tilde{B} \simeq \mathrm{Mat}_{\tilde{m}}(F[D])$ by Theorem 3.4 where $\tilde{m} = |N : C|m(b)$, and $Z(\tilde{B}) \simeq F[D]$.

Finally, we recall from Lemma 2.6 that Br_D induces a homomorphism from $Z(B)$ into $Z(\tilde{B})$. So we are done if we can prove that this is an isomorphism. By Lemma 2.6,

(41) $$Br_D(Z(B)) = \underline{\tilde{B}}(F[C])_D^N .$$

On the other hand, $Z(\tilde{B}) \simeq Z(b) = \underline{b}(F[C])_D^C$ by Corollary 1.6 and thus

(42) $$Z(\tilde{B}) = Tr_C^N(Z(b)) = Tr_C^N(\underline{b}F[C])_D^C \subseteq \underline{\tilde{B}}F[C]_D^N$$

and we are done.

Remark. Similarly, we get that the corresponding block $\hat{B}$ in $R[G]$ is isomorphic to $\mathrm{Mat}_m(R[D])$ in Lemma 8.25 and Theorem 8.26.

Remark. We point out that if $\mathbb{B}$ is a block of $G = DC_G(D)$ with D as defect group, then $\mathbb{B}$ has exactly one irreducible character with D in the kernel, namely the irreducible character in the corresponding block of G/D of defect 0. This character is called the canonical character of $\mathbb{B}$.

9. Vertices of simple modules.

In this section we prove a deep result due to Knörr (1979) on vertices of simple modules and R-forms of irreducible characters. Combining his ideas with the idea behind our proof of the extended first main theorem, we next derive a new result from which a result of Erdmann (1977) on simple modules with cyclic vertices follows.

We are already well prepared for the proof of Knörr's Theorem because of our discussion in Chapter II, Section 11.

Let (F, R, S) be a p-modular system, and let θ equal F or R.

Let $X \in M_\theta(G)$ be indecomposable, where G is arbitrary. The name of the game is the following: Rather than assuming that X (for $\theta = F$) or $X \otimes_R S$ (for $\theta = R$) is simple, we assume that if D is a vertex of X and $G_1 = N_G(D)$, then $(X,X)^{\mathfrak{X},G}$ is a cyclic θ-module, i.e. a homomorphic image of θ, where $\mathfrak{X} = \{D \cap D^g \mid g \notin G_1\}$. Then the same holds if we replace X by its Green correspondent $f(X)$ in G_1 and G by G_1, by Corollary II.5.8. Next, Corollary II.5.4 will allow us to go down to the inertial group T of a source N of $f(X)$, if we replace $\mathfrak{X}$ by $\mathfrak{Y} = \mathfrak{X} \cap_{G_1} T$. This is how far we will get with ideas already developped.

We must consider the following situation: T is a finite group, $D \trianglelefteq T$ is a p-group, and $N \in M_\theta(D)$ is indecomposable and T-stable. Set $C_0 = C_G(D)$ and $C = DC_0$. Next we set

$$(1) \qquad E_1 = (N,N)^D, \qquad E_C = (N^{\uparrow C},N^{\uparrow C}), \qquad E = (N^{\uparrow T},N^{\uparrow T}).$$

As we saw in Chapter II, Section 11, we then have the embeddings

(2) $$E_1 \xrightarrow{Fr_D^C} E_C \xrightarrow{Fr_C^T} E$$

Moreover, there is a natural embedding

(3) $$\varepsilon : R[C_0] \to E_C$$

namely the homomorphism

(4) $$\varepsilon(c) : \Sigma n_x \otimes x \to \Sigma n_x \otimes cx$$

Now, with the notation following Lemma II.11.4 we may choose ϕ_c as the identity for $c \in C_0$. It then follows that $\varepsilon(c) = \psi_c$.

Finally, we recall from our discussion following Proposition II.11.8 that

(5) $$F/J(E_1)E \simeq F^{\wedge}[T/D]$$
$$E_C/J(E_1)E_C \simeq F[C/D]$$

We now have the following commutative diagrams

(6) $$\begin{array}{ccccc} \theta[C_0] & \xrightarrow{\varepsilon} & E_C & \xrightarrow{Fr_C^T} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma_0} & & \downarrow{\scriptstyle\gamma} \\ \theta[C] & \xrightarrow[\beta]{} & F[C/D] & \xrightarrow[\rho]{} & F^{\wedge}[T/D] \end{array}$$

where γ, γ_0 and β are the canonical maps and α and ρ the embeddings.

Indeed that the first diagram commutes follows from the fact that $\varepsilon(c) = \psi_c$. Notice that $\psi_{c_1}\psi_{c_2} = \psi_{c_1 c_2}$ for $c_1, c_2 \in C$. That the second diagram commutes follows from the fact that

(7) $$J(E_1)E \cap E_C = J(E_1)E_C$$

and thus γ_0 is just the restriction of γ to E_C. Hence Fr_C^T induces a map from $F[C/D]$ into $F^{\wedge}[T/D]$ which is identical to the embedding, as we may deduce from our discussion prior to Proposition II.11.5.

Moreover, as the kernels of γ_0 and β are both contained in the radical, there is a one-to-one correspondence between the idempotents of $\mathcal{O}[C_0]$ and those of $F[C/D]$, since moreover $\beta \circ \alpha$ is surjective. Thus the same holds in relation to $\theta[C]$ and E_C.

We now have

<u>Proposition 9.1</u>. (Knörr (1979)) Let $e \in E$ be an idempotent such that $\gamma(e)\hat{F}[T/D]$ is semisimple. Then all components of $(e(N^{\uparrow T}))_{\downarrow C}$ lie in blocks in $\theta[C]$ with defect group D in C.

<u>Proof</u>: Recall that E is a free E_C-module and let $eE \simeq \oplus f_i E_C$ as an E_C-module for suitable primitive idempotents $f_i \in E_C$. Choose for all i $e_i \in \theta[C]$ a primitive idempotent with $\varepsilon(e_i) = f_i$. Then

$$(8) \qquad \gamma(e)\hat{F}[T/D] \simeq \bigoplus_i \gamma_0(f_i)F[C/D]$$

$$\simeq \bigoplus_i \gamma_0 \circ \varepsilon(e_i)F[C/D]$$

$$\simeq \bigoplus_i \beta \circ \alpha(e_i)F[C/D] .$$

By assumption, $\gamma(e)\, J(\hat{F}[T/D]) = 0$. As $C \vartriangle T$, this forces $\gamma_0(f_i)\, J(F[C/D]) = 0$ for all i and thus $\beta \circ \alpha(e_i)$ belongs to a block of defect 0. Hence $\alpha(e_i)$ belongs to a block with defect group D by Lemma 3.8. Finally,

$$(9) \qquad (e(N^{\uparrow T}))_{\downarrow C} = \bigoplus_i f_i(N^{\uparrow C})$$

$$= \bigoplus_i [\varepsilon(e_i)](N^{\uparrow C})$$

$$= \bigoplus_i \alpha(e_i)N^{\uparrow C}$$

by (6).

<u>Proposition 9.2</u>. (Knörr (1979)) Let $D \vartriangle G$ be a p-group and set $C = DC_G(D)$. Let B be a p-block of G and let X be an indecomposable B-module which is D-projective.

Assume there exists a family $\mathfrak{U}$ of subgroups of G such that

$0 \neq (X,X)^{\mathfrak{X},G}$ is a cyclic θ-module, and let b be a root of B in $C_G(D)$. Then b has defect group D in C.

<u>Proof</u>: Note that as $(X,X)^G/J((X,X)^G)$ is a division ring, and a cyclic θ-module by our assumption, the only possibility is $F = R/(\pi)$ In particular, X is absolutely indecomposable (see Appendix III).

Let $N \in M_\theta(D)$ be a source of X and let T be the inertial group of N. Let $M|N^{\uparrow T}$ be indecomposable with $X|M^{\uparrow G}$. Then actually, $X = M^{\uparrow G}$ by Lemma II.ii.4. Set $(M^{\uparrow G})_{\downarrow T} = M \oplus M_0$. Then

$$(10) \qquad (X,X)^{\mathfrak{X},G} \simeq (M,M)^{\mathfrak{X}\cap_G T,T} \oplus (M,M_0)^{\mathfrak{X}\cap_G T,T}$$

by Corollary II.5.4. Furthermore, M is $\mathfrak{X}\cap_G$ T-projective if the first term equals 0, by Theorem II.2.3 and Lemma II.3.9. But then the second term would be 0 too, a contradiction. Thus

$$(11) \qquad (X,X)^{\mathfrak{X},G} \simeq (M,M)^{\mathfrak{X}_1,T}$$

where $\mathfrak{X}_1 = \mathfrak{X}\cap_G T$. Now, let $e \in E = (N^{\uparrow T},M^{\uparrow T})^T$ be the idempotent corresponding to the direct summand M. Then $eEe \simeq (M,M)^T$ and

$$(12) \qquad eE_{\mathfrak{X}_1}e = e(E_1)_{\mathfrak{X}_1\cap_T D}Ee$$

by Proposition II.11.8, where $E_1 = (N,N)^D$. So

$$(13) \qquad (M,M)^{\mathfrak{X}_1,T} \simeq eEe/eE_{\mathfrak{X}_1}e \simeq eEe/e(E_1)_{\mathfrak{X}_1\cap_T D}{}^{DEe}$$

is cyclic. As $(E_1)_{\mathfrak{X}_1\cap_T D} \subseteq J(E_1)$, this forces

$$(14) \qquad \gamma(e)F^{\wedge}[T/D]\gamma(e) \simeq F$$

with the notation of (6). Thus $\gamma(e)F^{\wedge}[T/D]$ is simple, as $F^{\wedge}[T/D]$ is a symmetric algebra. Finally, as all direct summands of $M_{\downarrow C}$ lie in roots of B by Lemma 3.2, we are done by Proposition 9.1.

Using the same notation, this proof combined with the idea behind the proof of Theorem 4.1 moreover yields

Proposition 9.3. Same notation and assumptions as above. Assume furthermore that S is a splitting field of $S[C/D]$. Let $Y|X_{\downarrow C}$ be indecomposable with vertex D such that $X|Y^{\uparrow G}$, and let T_1 be inertial group of Y in $N_G(D)$. Then $|T_1/C|$ is prime to p. In particular, if Y lies in the block b of $F[C]$, the defect group of b in T_1 is D.

Proof: We may as well assume that $M|Y^{\uparrow T}$. Choose $M_1|Y^{\uparrow T_1}$ with $M = M_1^{\uparrow T}$. Then by the same argument as above,

$$(15) \qquad (M,M)^{\mathfrak{U}_1, T} \simeq (M_1, M_1)^{\mathfrak{U}_1 \cap_T T_1, T_1}$$

and thus all statements above hold if we replace T by T_1. In particular, if $e_1 \in (Y^{T_1}, Y^{T_1})$ is the idempotent corresponding to the direct summand M_1 of Y^{T_1}, the corresponding idempotent $\gamma(e_1) \in F^{\wedge}[T_1/D]$ has the property that $\gamma(e_1)F^{\wedge}[T_1/D]$ is simple. Now let $\bar{b}$ be the block of $F[C/D]$ corresponding to b. Obviously, $T_1 \leq N(b)$. We now claim that the corresponding idempotent $\underline{\bar{b}}$ is a block idempotent in $F^{\wedge}[T_1/D]$. Let $x \in T_1$. Then

$$(16) \qquad \alpha(t,t^{-1})^{-1} t^{-1} \cdot \underline{\bar{b}} \cdot t$$

is a block idempotent of $F[C/D]$. However, as it contains the module $Y_1 \otimes t \simeq Y_1$, it must be equal to $\underline{\bar{b}}$, or, in other words, $t \cdot \underline{\bar{b}} = \underline{\bar{b}} \cdot t$ as claimed. Moreover, $\gamma(e_1)$ above lies in this block, which shows that $\underline{\bar{b}}F^{\wedge}[T_1/D]$ is a simple algebra. Now, let $D \leq Q \leq T$ with $|T : Q|$ prime to p. Then $\underline{\bar{b}}F^{\wedge}[Q/D]$ is a simple algebra as well, as $\underline{\bar{b}} \in F[C/D] \subseteq F[Q/D]$.

Finally, let A be the simple $\bar{b}$-module. Then the p-part of $\dim_F A$ is the order of a Sylow p-subgroup of C/D. Moreover, $\hat{A} = AF^{\wedge}[Q/D]$ is semisimple and projective, and the p-part of its dimension is precisely that of $\dim_F A$ times $|Q/C|$. However, any direct summand of $\hat{A}$ has a dimension divisible by this number as well, by Proposition II.11.8. Finally, as $\hat{A}_{\downarrow F[C/D]} \simeq A^{(|Q/D|)}$ we deduce just as in the case of the extended first main theorem that $\hat{A}$ is simple. But this is clearly a contradiction, as C is normal in Q, unless $C = Q$.

We may now prove our main results. First, we have Knörr's Theorem.

Theorem 9.4. Let $X \in \mathbb{M}_\theta(G)$ be indecomposable in the block B and let D be a vertex of X. Set $\mathfrak{X} = \{D \cap D^g | g \notin N_G(D)\}$, and assume $(X,X)^{\mathfrak{X},G}$ is a cyclic θ-module. Then there exists a B-subpair (D,b).

Proof: Let f denote Green correspondence between G and $H = N_G(D)$ w.r.t. D. Then $(f(X),f(X))^{\mathfrak{X},H}$ is cyclic by Corollary II.5.8. Let B_1 be the block containing $f(X)$, set $C = DC_G(D)$ and let b be any root of B_1 in C. Then D is a defect group of b by Proposition 9.2. As $Br_D(\underline{B})B_1 \neq 0$ by Theorem 5.1 b is a root of B as well, and we are done.

Corollary 9.5. Same notation as above. Assume $\theta = R$ and that M is an R-form of an irreducible character. Then there exists a B-subpair (D,b).

Proof: If $M \otimes_R S$ is simple, then $(M,M)^G \simeq R$ and thus the assumption of Theorem 9.3 holds.

Corollary 9.6. Same notation as in Theorem 9.3. Assume $\theta = F$ and that M is a simple F[G]-module. Then there exists a B-subpair (b,D).

Proof: Same argument.

Corollary 9.7. Let $\mathbb{B}$ be a p-block of G. Let V be the vertex of a simple F[G]-module in $\mathbb{B}$ or an R-form of an irreducible character in $\mathbb{B}$. Then there exists a defect group D of B in G such that

$$C_D(V) \leq V \leq D. \tag{17}$$

Proof: By the corollaries above and Theorem 3.3 iii).

Another consequence of Theorem 9.4 is of course that $O_p(C) = D$. However, Proposition 9.3 allows us to improve this considerably.

Theorem 9.8. Same notation and assumption as in Theorem 9.4. Let $Y | X_{\downarrow DC_G(D)}$ be indecomposable with $X | Y^{\uparrow G}$, and let T_Y be the inertial group of Y in $N_G(D)$. Assume furthermore that S is a splitting field of $C_G(D)$.. Then $|T_Y : DC_G(D)|$ is prime to p.

Proof: Just as in Theorem 9.4, Corollary II.5.8 reduces the problem to the case where D is normal in G.

Corollary 9.9. (K. Erdmann (1977b)) Let B be a block of F[G] and let X be a simple B-module with cyclic vertex D. Then D is a defect group of B.

Proof: We may assume that F is a splitting field of $C_G(D)$ (see appendix III). Let b be the block of $F[C_G(D)]$ containing Y with the notation of Theorem 9.8. By Theorem 8.26 and the example of Chapter II, Section 8, the p.i.m. of b is uniserial and Y is a submodule. Hence $Y \otimes g \simeq Y$ whenever $g^{-1}bg = b$ and $T_Y = N(b)$. By the Extended First Main Theorem, B has D as defect group.

Remark. A similar result does not hold if we replace F by θ and assume X is the R-form of an irreducible character. The point is that we cannot necessarily deduce from $g^{-1}bg = b$ that $Y \otimes g \simeq Y$.

As an example, let $\langle x \rangle$ be a cyclic subgroup of order 4 in a dihedral group P of order 8. Let ζ be a faithful irreducible character of $\langle x \rangle$. Then $\zeta^{\uparrow P}$ is irreducible, simply because the inertial group of ζ is $\langle x \rangle$, which is also the vertex of an R-form of ζ.

As the point of the proof above is to deduce that $Y \otimes g \simeq Y$ whenever $g^{-1}bg = b$, it is natural to ask when this happens. Thus we introduce

Definition 9.10. Let P be a p.i.m. of F[G] and let M be a submodule of P. Then M is called characteristic in P (see Brandt (1981)) if $\phi(M) \simeq M$ for all $\phi \in (M,M)^G$.

It follows that whenever Y is characteristic in the p.i.m. of b, then D is a defect group of B.

For a number of results on characteristic submodules, see Brandt (1981) and (1983).

It is quite feasable that the results above may be considerably improved or supplemented. For a discussion of this, see Willems (1981).

10. Defect groups.

In this final section, we shall make use of the theory of permutation modules we developed in Chapter II, Section 12.

We start with a new proof of a result of Green, which states that a defect group of a block is always the intersection of two Sylow p-subgroups.

Proposition 10.1. Let $Q \in \mathrm{Syl}_p(G)$ and let I_Q denote the trivial F[Q]-module. Let B be an arbitrary block of G, and let D be a defect group of B. Then $I_Q^{\uparrow G}$ has a direct indecomposable summand in B with D as a vertex.

Proof: (Sibley) By Theorem II.3.10 and Theorem 5.1, it suffices to prove that $(I_Q^{\uparrow G})_{\downarrow N_G(D)}$ has a direct summand in the block $\tilde{B}$ of $F[N_G(D)]$ corresponding to B with D as a vertex. By Mackey decomposition, it therefore suffices to prove that for some $\gamma \in G$,

(1) $$\left((I_Q \otimes \gamma)_{\downarrow Q^{\gamma} \cap N_G(D)}\right)^{\uparrow N_G(D)}$$

has a direct summand in $\tilde{B}$ with that property. Choosing $\gamma = 1$ we conclude that it suffices to prove the statement when D is normal in G. As D is in the kernel of I_Q it therefore suffices to consider the case where $D = 1$, or in other words B is of defect 0. But this case follows immediately from the Nakayama relations. Indeed, let M be the simple B-module. Then $(I_Q^{\uparrow G}, M)^G \simeq (I_Q, M)^Q \neq 0$ as I_Q is the only simple F[Q]-module. As M is projective, it follows that $M \mid I_Q^{\uparrow G}$.

Corollary 10.2. (Green (1968)) Same notation as above. Then $D = Q \cap Q^{\gamma}$ for some $\gamma \in G$.

Proof: By Lemma II.12.5.

Independently of this, let us make another connection to trivial source modules.

Lemma 10.3. Let B be a block of $F[G]$ with D as defect group, and let $\tilde{B}$ be the corresponding block of $F[N_G(D)]$. Then all simple modules in $\tilde{B}$ are trivial source modules with the trivial $F[D]$-module I_D as source.

In particular, their Green correspondents in B are liftable, as the $F[G]$-homomorphisms between them.

Proof: Any simple $\tilde{B}$-module has D in its kernel and is projective as an $F[N(D)/D]$-module, hence is a trivial source module. Also, the Green correspondent lies in B by Theorem 5.1. The second part now follows from Theorem II.12.4.

We proceed to briefly discuss the idea of lower defect groups, inspired by Burry (1982), and refer to the very recent Green (1983) for further developments. For different approaches, see Iizuka (1972), Broué (1979) and Olsson (1980). The original idea goes back to Brauer (1969).

Let $\mathcal{K}_1,\dots,\mathcal{K}_k$ denote the conjugacy classes in G and choose $x_i \in \mathcal{K}_i$ for all i. Set $C_i = C_G(x_i)$ and for any $H \leq G$, the trivial 1-dimensional $\theta[H]$-module is denoted by I_H, where θ is F or R for (F,R,S) a p-modular system.

The whole point is now to let $A = \theta[G]$ act on itself by conjugation, which gives A a different structure as an A-module than the usual.

We observe that the blocks $\{B_i\}$ of A are invariant under this action and

(2) $$A \simeq \bigoplus_{i=1}^{k} I_{C_i}^{\uparrow G} .$$

Thus each block is isomorphic to a direct sum of certain trivial source modules.

In Lemma II.12.7, we saw that $I_{C_i}^{\uparrow G}$ has exactly one indecomposable direct summand $P(C_i)$, which contains I_G as a sub- or factor-module, and if $Q_i \in \mathrm{Syl}_p(C_i)$, then Q_i is a vertex of $P(C_i)$.

Moreover, $P(C_i)$ is entirely determined by Q_i as we saw in Lemma 12.7 (these modules are called Scott-modules in Burry (1982)).

Notation. Following Burry (1982), we let $\mathcal{P}(G)$ denote a complete set of representatives of the conjugacy classes of p-subgroups of G and let B be a block of $F[G]$. For $Q \in \mathcal{P}(G)$, the multiplicity of $P(Q)$, in the notation of Lemma II.12.7) as a direct summand of B under conjugate action is denoted by $m_B(Q)$.

We now have the following beautiful result.

Theorem 10.4. (Burry (1982)) The multiplicity of Q as a lower defect group of B (see Definition 2.7) is equal to the multiplicity above.

Proof: We shall give a very short proof which is partly inspired by Green (1983), although more direct.

Let $K_i = \mathrm{Span}_F\{g \mid g \in \mathcal{K}_i\}$. Then K_i is a right $F[G]$-module under conjugate action, and as such isomorphic to $I_{C_i}^{\uparrow G}$. As we have discussed above, $P(C_i)$ is a direct summand of this module, and its trivial submodule is obviously spanned by $v_i = [\mathcal{K}_i]$. Now for $Q \in \mathcal{P}(G)$ it follows that

$$A_Q^G = \mathrm{Span}_F\{v_i \mid Q_i \underset{G}{\leq} Q\}$$

(3) $$A_{<Q}^G = \mathrm{Span}_F\{v_i \mid Q_i \underset{G}{<} Q\}$$

$$A_{=Q}^G = \mathrm{Span}_F\{v_i \mid Q_i \underset{G}{=} Q\}$$

We now take advantage of the facts that the two former are ideals in $Z(G)$ and that (see Section 2, (7))

(4) $$\dim_F A_{=Q}^G = \dim_F(A_Q^G) - \dim_F(A_{<Q}^G)$$

is the multiplicity of $P(Q)$ as a direct summand of $A = F[G]$ under conjugate action. Indeed the fact that A_Q^G and $A_{<Q}^G$ are ideals imply that

(5) $$A_Q^G = \bigoplus_i A_{Q}^{G}B_i, \qquad A_{<Q}^G = \bigoplus_i A_{<Q}^{G}B_i$$

where B_i runs through the blocks of $F[G]$. Hence

(6) $$\dim_F(A_Q^G B) = \sum_{R \underset{G}{\leq} Q} m_B(R)$$
$$\dim_F(A_{<Q}^G B) = \sum_{R \underset{G}{<} Q} m_B(R)$$

and as $m_B(Q) = \sum_{R \underset{G}{\leq} Q} m_B(R) - \sum_{R \underset{G}{<} Q} m_B(R)$, we are done by (4) and Definition 2.7.

We may now use Burry's characterization of the multiplicity of lower defect groups to derive a number of the fundamental properties in a very easy and straightforward manner.

Lemma 10.5. (Brauer (1969)) Let D be a defect group of B and Q a p-subgroup of G. Then

i) $m_B(Q) = 0$ if $Q \underset{G}{\not\leq} D$

ii) $m_B(D) \neq 0$.

Proof: The vertex of $P(Q)$ is Q. As $P(Q)$ is D-projective by Lemma 1.15 if $P(Q)$ is a B-module, i) follows

ii) is by definition and Lemma 1.7.

Theorem 10.6. (Brauer (1969)) Let $\{B_i\}$ denote the blocks of $F[G]$. Then

(7) $$\sum_{Q \in \mathcal{P}(G)} m_B(Q) = \dim_F Z(B)$$

and

(8) $$\sum_{B_i} m_{B_i}(Q) = |\{\mathcal{K}_j \mid D(\mathcal{K}_j) \underset{G}{=} Q\}|.$$

Proof: Under conjugate action, $Z(B) = B^G$ while K_i^G is 1-dimensional, where $K_i = \mathrm{Span}_F\{g \mid g \in \mathcal{K}_i\}$, from which (7) follows. Moreover, as $F[G] \simeq \oplus K_i$ under conjugate action and $D(\mathcal{K}_i)$ is a vertex of the direct summand containing the trivial submodule, (8) follows.

The next result shows that the lower defect groups, just as the defect groups, can be determined locally.

Theorem 10.7. (Brauer (1969)) Let $\{b_i\}$ be the blocks of $F[N_G(Q)]$ such that $Br_Q(\underline{B})\underline{b}_i \neq 0$. Then

(9) $$m_B(Q) = \sum_i m_{b_i}(Q).$$

Proof: By Lemma 2.6 and Definition 2.7, as $F[C_G(Q)]_Q^{N_G(Q)} = F[N_G(Q)]_{=Q}^{N_G(Q)}$ with the notation we used there.

Remark. Theorem 8.26 demonstrates the importance of lower defect groups. Indeed, Lemma 2.6 iii) which was the key to Theorem 8.26, can be restated:

Lemma 10.8. (Olsson (1980)). The multiplicity of D as a lower defect group of B is equal to $\dim_F Br_D(Z(D))$.

Finally the theory of lower defect groups offer an interesting explanation of the elementary divisors of the Cartan matrix of a block.

We continue to let $\{\mathcal{K}_i\}$, $i=1,\ldots,k$ denote the conjugacy classes of G such that the first ℓ classes are those consisting of p-regular elements. Set

(10) $$Z_\theta^1(G) = \mathrm{Span}_\theta\{[\mathcal{K}_i] \mid i=1,\ldots,\ell\} \subseteq \theta[G].$$

If $K_i = \mathrm{Span}_\theta\{g \mid g \in \mathcal{K}_i\} \leq \theta[G]$ and we let $\theta[G]$ act on $A = \theta[G]$ as before, then $Z_\theta^1(G) = (A^1)^G$, where $A^1 = \bigoplus_{i=1}^{\ell} K_i$.

Assume in the following that S is a splitting field of G and let $\phi_1,\ldots,\phi_\ell$ resp. $\Phi_1,\ldots,\Phi_\ell$ denote the irreducible Brauer resp. indecomposable projective characters, and choose notation so that the first ℓ_B of those lie in the p-block $\mathbb{B}$ of G, where B is the corresponding block of $R[G]$.

Choosing $\theta = R$ in the following, we have a natural R-isomorphism

(11) $$\Gamma : (A^1, R) \to A^1$$

given by $\eta \to \sum_{g \in G_0} \eta(g^{-1})g$. Observe that under the R[G]-action induced on (A^1, R),

(12) $$(A^1, R)^G = \mathrm{Span}_R\{\phi_1, \ldots, \phi_\ell\}.$$

Thus Γ induces an R-isomorphism

(13) $$\Gamma : \mathrm{Span}_R\{\phi_1, \ldots, \phi_{\ell_B}\} \to Z^0_R(B)$$

where $Z^1_R(B) = Z^1_R(G) \cap B$.

Let $\lambda : R[G] \to R$ be the augmentation map, e.e., $\lambda(\Sigma\alpha_g g) = \alpha_1$, and set $\lambda_a(b) = \lambda(ba)$ for all $a, b \in R[G]$. Then $(A^1, R) = \mathrm{Span}_R\{\lambda_g \mid g \in G_0\}$.

<u>Proposition 10.9</u>. (Broué (1979)) With the notation above,

i) $\Gamma((\underline{B}A^1, R)^G_1) = (\underline{B}A^1)^G_1$

ii) $(\underline{B}A^1, R)^G_1 = \mathrm{Span}_R\{\Phi_1, \ldots, \Phi_{\ell_B}\}$

iii) $(\underline{B}A^1)^{1,G} \simeq \mathrm{Span}_R\{\phi_1, \ldots, \phi_{\ell_B}\}/\mathrm{Span}_R\{\Phi_1, \ldots, \Phi_{\ell_B}\}$.

<u>Proof</u>: i) is by definition as $\Gamma(\eta g) = \Gamma(\eta)g^{-1}$. To prove ii) and iii) we first show that $\Phi_i \in (\underline{B}A^1, R)^G_1$ for all $i = 1, \ldots, \ell_B$. To see this let $x_j \in \mathcal{K}_j$. It then follows from Theorem I.15.9 iii)c) and Lemma 8.2 that

(14) $$\frac{\Phi_i(x_j)}{|C_G(x_j)|} \in R$$

for all i,j. Define $\lambda_i \in (\underline{B}A^1, k)$ by

(15) $$\lambda_i = \sum_{j=1}^{\lambda_0} \frac{\Phi_i(x_j)}{|C_G(x_j)|} \lambda x_j .$$

Then $\Phi_i = \sum_{y \in G} \lambda_i g$ and thus $\Phi_i \in (\underline{B}A^1, R)^G$.

Thus ii) will in fact follow from i) and iii) which in turn will follow if we can prove

(16) $$(A^1)^{1,G} \simeq \mathrm{Span}_R\{\phi_1,\dots,\phi_\ell\}/\mathrm{Span}_R\{\hat\phi_1,\dots,\hat\phi_\ell\}.$$

However, from Corollary 8.7 we know that the right hand side is isomorphic to

(17) $$\mathbf{Z}_{|D(\mathcal{K}_1)|} \oplus \dots \oplus \mathbf{Z}_{|D(\mathcal{K}_\ell)|} .$$

But so is the left hand side, as

(18) $$\{|C_G(x_1)|[\mathcal{K}_1],\dots,|C_G(x_\ell)|[\mathcal{K}_\ell]\}$$

is a basis of $(A^1)^G_1$.

Notation. Inspired by Theorem 10.4, let for $Q \in \mathcal{P}(G)$ the number $m^1_B(Q)$ denote the multiplicity of $\hat{P}(Q)$ as a direct summand of $\underline{B}A^1$, where $\hat{P}(Q)$ is the R[G]-module corresponding to P(Q).

Also, we set

(19) $$Z^1_m(B) = \underline{B} \sum_{|Q| \le p^m} (A^1)^G_Q .$$

As a corollary of Proposition 10.8, we now have

Theorem 10.9. The following numbers are equal

i) $a_1 = \mathrm{rank}_R(Z^1_m(B)/Z^1_{m-1}(B))$

ii) $a_2 = \sum_{|Q| \le p^m} m^1_B(Q)$

iii) The multiplicity a_3 of p^m as an elementary divisor of the Cartan matrix of B.

Remark: That $a_1 = a_3$ goes back to Brauer.

Proof: That $a_1 = a_3$ follows directly from Proposition 10.8 and that $a_1 = a_2$ is by the same argument as that used to prove Theorem 10.4.

As the last topic we discuss the second question raised in Section 3. How does one determine the blocks of a group of defect 0 ? We have already pointed out that this is completely decided by the character table, but it would be desirable to have a more ring theoretic characterization of such blocks, such as

Theorem 10.10. (Tsushima (1971b)) Let s_1 be the sum of all p-elements in G. Then s_1^2 equals the sum of all block idempotents of defect 0. In particular, the number of such blocks is equal to $\dim_F(s_1 Z(F[G]))$.

Before we embark on the proof, we recall some facts. Denote the radical of F[G] by J. Recall (Lemma I.8.3) that the annihilator of J is the socle S_1 of F[G] as an F[G]-module.

If F is algebraically closed and $E_1,\dots,E_\ell$ denote the simple F[G]-modules, then $F[G]/J \simeq \bigoplus_{i=1}^{\ell} A_i$ as an algebra, where $A_i \simeq (E_i,E_i) \simeq \mathrm{Mat}_{n_i}(F)$ with $n_i = \dim_F E$. Thus the canonical homomorphism $F[G] \to A_i$ gives rise to a matrix representation corresponding to the simple module. We then define $t_i : F[G] \to F$ as follows: For $x \in F[G]$, denote the corresponding matrix in A_i, using the isomorphism above, by $\underline{\underline{X}}$, and set $t_i(x) = \mathrm{tr}(\underline{\underline{X}})$, the sum of all diagonal elements in $\underline{\underline{X}}$. Note that if x is a p-regular element and ϕ_i is the Brauer character defined by E_i, then

$$(20) \qquad t_i(x) = \bar{\phi}_i(x) := \phi_i(x) + (\pi).$$

Thus $t_1,\dots,t_\ell$ are linearly independent as class functions on G_0, the p-regular elements of G, by Corollary 8.3.

Following Tsushima (1971a) we now have

Lemma 10.11. With the notation above,

i) $\sum_{g \in G} t_i(g^{-1})g \in S_1$ for all i.

ii) $s_1 \in S_1$.

Proof: i) Let $\lambda \in F[G]^*$ denote the augmentation map $\Sigma\alpha_g g \to \alpha_1$, and set $\lambda_a(x) = \lambda(xa)$ for all $x \in F[G]$. Then $t_i = \lambda_{c_i}$ for some $c_i \in F[G]$ by Theorem I.6.3. Moreover, if $c_i = \Sigma\alpha_g g$,

$$\alpha_g = \lambda(c_i g^{-1}) = \lambda(g^{-1}c_i) = t_i(g^{-1}) \tag{21}$$

and thus $c_i = \Sigma t_i(g^{-1})g$. Finally, as $0 = t_i(J) = \lambda(Jc_i)$, $Js_i = 0$ by Lemma I.6.10 and thus c_i S_1.

ii) As $t_1,\ldots,t_\ell$ are linearly independent on G_0, there exists $\gamma_1,\ldots,\gamma_\ell \in F$ such that $\Sigma\gamma_i t_i(x) = 0$ for $x \in G_0\backslash\{1\}$, while $\Sigma\gamma_i t_i(1) \neq 0$. Furthermore, if $g = g_p g'$ is the p-decomposition of g, we have that $t_i(g) = t_i(g')$. Now let $1=x_1,\ldots,x_\ell$ be representatives of the p-regular conjugacy classes, and let s_j be the sum of all elements in G, whose p'-part is x_j. Then $c_i = \sum_{j=1}^{\ell} t_i(x_j)s_j$, and it follows that

$$\sum_{i=1}^{\ell} \gamma_i c_i = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} \gamma_i t_i(x_j)c_j = \Sigma\gamma_i t_i(1)s_1 \tag{22}$$

and we are done.

Remark: A similar argument shows of course that $s_i \in S_1$ for all i. The ideal generated by the s_i 's is called Reynolds' ideal, and it is now an easy exercise to prove that Reynolds' ideal equals S_1. For more on this, see Reynolds (1972), O'Reilly (1974) and Olsson (1980).

Proof of Theorem 10.10: If $\overline{F}$ is the algebraic closure of F, then $S_1(\overline{F}[G]) = S_1(F[G]) \otimes_F \overline{F}$, and if $e \in F[G]$ is a block idempotent with D as defect group then e is a sum of block idempotents in $\overline{F}[G]$ with D as defect group. Thus it suffices to prove the theorem in the case where F is algebraically closed.

Let $\hat{s}_1$ be the corresponding element in $R[G]$ and let $1 = \Sigma\varepsilon_i' + \Sigma\varepsilon_j''$ be a centrally primitive idempotent decomposition in $R[G]$ where the ε_i' 's all have non-trivial defect groups while the ε_i'' 's all have trivial defect groups. Then the ε_i'' 's are precisely those idempotents in the decomposition above which remains primitive in $S[G]$. Let χ_j'' be the irreducible character corresponding to ε_j''. Then $\hat{s}_1 e_j'' = \omega_j''(\hat{s}_1)e_j''$, where ω_j'' is the corresponding central character, as

$\hat{s}_1 \in Z(S[G])$. However, as χ''_j is in a block of defect 0, ω''_j is zero on any class sum of proper p-elements and thus $\omega''_j(\hat{s}_1) = 1$. It follows that

(23) $$s_1 = s_1(\Sigma\varepsilon'_i + \Sigma\varepsilon''_j$$

and thus

(24) $$s_1^2 = s_1^2(\Sigma\varepsilon'_i) + \Sigma\varepsilon''_j .$$

However, $s_1(s_1(\Sigma\varepsilon'_i)) \in S_1 \cdot J = 0$ and thus

(25) $$s_1^2 = \Sigma\varepsilon''_j$$

Recently, Robinson (1983) has added an interesting contribution to the problem of deciding the existence of p-blocks which we will discuss briefly. For other results in this direction, see Willems (1978).

Let $P \in Syl_p(G)$ and let D be a normal p-subgroup of G where we include the case $D = 1$. Let $\{x_1,\ldots,x_r\}$ be a complete set of representatives of those conjugacy classes of p-regular elements in G which have defect group D. Set $P\,G/P = \{g_j\}_1^n$, where the g_j's are chosen wherevery possible to satisfy simultaneously

i) g_j is p-regular

ii) $D \in Syl_p(C_G(g_j)$

iii) $P \cap P^{g_j} = D$.

Now we choose notation so that the first m g_j's satisfy condition i), ii) and iii) and the rest not. If $m \neq 0$, we define an $r \times m$ matrix N with entries in $GF(p)$ as follows: for $i=1,\ldots,r$, $j=1,\ldots,m$, n_{ij} is the residue mod p of the number of conjugates of x_i in $g_jC_P(D)$. If $m = 0$, we set $N = \{0\}$. Then

<u>Theorem 10.12</u>. (Robinson (1983)) The number of blocks of G with D as defect group is the rank of the matrix NN^T.

For a proof, see Robinson (1983).

Note that this extends Corollary 10.2. A striking consequence of Robinson's Theorem is that if $p = 2$ and P is T. I in G (i.e. $P \cap P^g = 1$ for $g \in G$, $g \notin N_G(P)$) then G has a block of defect 0 if G has a p-regular element with 1 as defect group.

It would be interesting if these ideas would eventually yield a direct proof of the following interesting question.

Let G be a finite group with a cyclic Sylow p-subgroup Q which is T. I. Then one of the following holds

i) $Q \Delta P$

ii) $F[G]$ has a block of defect 0.

Recently, Michler (1983) has proved that in a minimal counter example, $P \simeq \mathbf{Z}_p$ and self-centralizing, and G is simple. Using the classification of the simple groups, he is then able to reach a contradiction, thus providing an affermative answer to the question.

Appendix I: Extensions.

We will here briefly explain, why the important group defined in Chapter I, Section 10 (or vector space if we work with a group algebra) is called Ext. In fact, this has already been suggested by Lemma I.10.7. But first we mention the induced long exact sequences for Ext.

Given a short exact sequence of A-modules for some ring A

(1) $$0 \to X \overset{i}{\to} Y \overset{\varepsilon}{\to} Z \to 0$$

we recall the fundamental fact that for any A-module M, the induced sequences

(2) $$0 \to (N,X)^A \overset{i^*}{\to} (N,Y)^A \overset{\varepsilon^*}{\to} (N,Z)^A$$

(3) $$0 \to (Z,N)^A \overset{i_*}{\to} (Y,N)^A \overset{\varepsilon_*}{\to} (X,N)^A$$

are exact as well. This is part of the following.

Proposition 1. (Long exact sequences of Ext.) Given (1), there exists natural exact sequences

(4) $$0 \to (N,X)^A \overset{i^*}{\to} (N,Y)^A \overset{\varepsilon^*}{\to} (N,Z)^A \to \mathrm{Ext}_A^1(N,X) \to \mathrm{Ext}_A^1(N,Y)$$

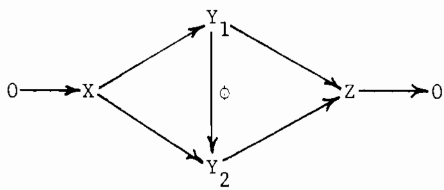
$$\to \mathrm{Ext}_A^1(N,Z) \to \mathrm{Ext}_A^2(N,X) \to \ldots$$

and similar starting with (3).

For the proof of this, we refer the reader to Rotmann (1979) or Hilton and Stambach (1971).

Let us now briefly discuss why the notation Ext has been chosen. For details see Rotmann (1979), which is the inspiration for the following (see also Curtis and Reiner (1981).

Given the exact sequence (1), the module Y (or the whole sequence) is called an <u>extension</u> of Z by X. Two such extensions are called <u>equivalent</u> if there is a commutative diagram

(5) 

Note that ϕ is necessarily an isomorphism (this is a consequence of The Five Lemma in general, but trivial if A is a finite dimensional algebra) and consequently this defines an equivalence relation. Thus equivalent extensions have isomorphic "middle" modules. The converse is not always true.

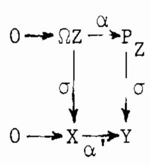
Now assume for convenience that the projective cover of an A-module is well-defined, and denote the projective cover of Z by P_Z. Then

(6) $$0 \to \Omega Z \overset{\alpha}{\to} P_Z \overset{\beta}{\to} Z \to 0$$

induces for any module X,

(7) $$0 \to (Z,X)^A \to (P_Z,X)^A \to (\Omega Z,X)^A \to \mathrm{Ext}_A^1(Z,X) \to 0$$

by definition, as $\mathrm{Ext}_A^1(P_Z,X) = 0$. For any $\sigma \in (\Omega Z,X)^A$ we next form the pushout Y of the pairs (α,σ),

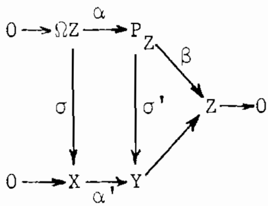
(8) 

As the reader may recall, this is done in the following way: In $X \oplus P_Z$, define

(9) $$W = \{\sigma(z) - \alpha(z) \mid x \in \Omega Z\}.$$

Now set $Y = (X \oplus P_Z)/W$ and define $\sigma' : P_Z \to Y$ by $\sigma'(x) := (x,0) + W$ and $\alpha' : X \to Y$ by $\alpha'(z) = (0,z) + W$. We then obtain a commutative diagram

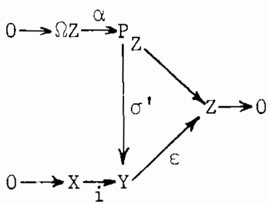
(10)



Thus each $\sigma \in (\Omega Z,X)^A$ determines an extension of Z by X. It is now easy to prove that σ_1 and σ_2 in $(\Omega Z,Y)^A$ determine equivalent extensions if and only if $\sigma_1-\sigma_2 \in \alpha^*((P_Z,X)^A)$. Thus each element of $\mathrm{Ext}_A^1(Z,X)$ determines an equivalence class of extensions of Z by X uniquely.

Conversely, let (1) be given. Then there exists $\sigma' \in (P_Z,Y)^A$ such that

(11)



commute, as P_Z is projective. As i is injective, this defines $\sigma \in (\Omega Z,X)^A$ such that $i \circ \sigma = \sigma' \circ \alpha$. Thus the extension (1) may be obtained as above from σ.

Appendix II: Tor.

Here we explain why the Tor groups are not defined or used in this book. This may be superfluous as the reader who already knows Tor will be aware of the explanation for that while the reader who is not will notice the conclusion which is that he or she may just as well remain in the dark as far as modules over group algebras are concerned.

For the first definition, let θ be a principal ideal domain, and let $X, Y \in \mathbb{M}_\theta(G)$. Recall that the tensor product of X and Y over the ring $\theta[G]$ is defined as

(1) $$X \otimes_G Y = X \otimes Y/\mathrm{Span}_\theta\{xg \otimes y - x \otimes yg\}.$$

In other words,

(2) $$X \otimes_G Y = X \otimes Y/((X \otimes Y)A(G))$$

where $A(G)$ is the augmentation ideal of $\theta[G]$. Thus $X \otimes_G Y$ is merely the maximal $\theta[G]$-factor module with trivial G-action.

<u>Definition 1</u>. $\mathrm{Tor}_1^G(X,Y)$ is defined by

(3) $$0 \to \mathrm{Tor}_1^G(X,Y) \to \Omega X \otimes_G Y \to P_X \otimes_G Y \to X \otimes_G Y \to 0$$

where P_X is the projective cover of X if this is well-defined or otherwise in general P_X is any projective module mapping onto X with ΩX defined as the kernel of this map.

To make it as short and painless as possible, assume now that $\theta = F$ is a field. Thus

$$(X \otimes_G Y, F) \simeq (X \otimes Y, F)^G \tag{4}$$

where F is considered as the trivial F[G]-module.

<u>Lemma 2</u>. There is a natural isomorphism

$$(\mathrm{Tor}_1^G(X,Y),F) \simeq \mathrm{Ext}_G^1(A,B^*). \tag{5}$$

In particular, $\mathrm{Tor}_1^G(X,Y) \simeq \mathrm{Ext}_1^G(X,Y^*)$ as F-spaces.

<u>Proof</u>: By Definition 1 and Lemma II.2.7,

$$\begin{aligned}(\mathrm{Tor}_1^G(X,Y),F) &\simeq (\Omega X \otimes Y,F)^{1,G} \\ &\simeq (\Omega X,Y^*)^{1,G}\end{aligned}$$

by Theorem II.6.10

$$\simeq \mathrm{Ext}_G^1(X,Y)^{1,G}$$

by Corollary II.2.6.

Appendix III: Extensions of the ring of coefficients.

It has been a deliberate aim to avoid assumptions on the p-modular system (F,R,S) supplying the rings of coefficients whenever possible and to avoid the discussion of what happens when the ring of coefficients is extended.

Sometimes though, a proof becomes much more transparent by passing to a splitting field which allows us to take advantage of the orthogonality relations. For completeness, let us recall that if A is a finite dimensional algebra over the field K, then K is called a splitting field of A if the Wedderburn components of A/J(A) are all matrix algebras over K and not just some division ring containing K.

For an excellent discussion of various important aspects of passing from a smaller field to a larger, we refer the reader to Chapter 9 of Isaacs (1976). In particular, we refer to Theorem (9.21) for a proof of the fact that in characteristic p, Schur indices are always 1.

Following Feit (1982), Chapter I, Section 18, we call $\hat{R}$ an extension of R if $\hat{R}$ is a principal ideal domain and local, free as an R-module and $J(\hat{R})^e = \hat{R}\pi$ for some e, the socalled ramification index of $\hat{R}$.

Let θ equal R or F. Then an indecomposable $\theta[G]$-module M is called absolutely indecomposable if for any finite extension $\hat{\theta}$ of θ, $M \otimes_\theta \hat{\theta}$ is indecomposable.

We have taken advantage of the following useful criterion for absolute indecomposability.

Lemma 1. Let $M \in \mathbf{M}_\theta(G)$ be indecomposable. Assume $(M,M)^G/J((M,M)^G) \simeq F$. Then M is absolutely indecomposable.

Proof: (See Feit (1982), p.72). Set $E = (M,M)^G$ and $\hat{E} = E \otimes_{\theta} \hat{\theta} \simeq (M \otimes_{\theta} \hat{\theta}, M \otimes_{\theta} \hat{\theta})^G$. Then $J(\theta)\hat{\theta} \subseteq J(\hat{\theta}) \subseteq J(\hat{E})$. In particular, $J(\theta)\hat{E} \subseteq J(\hat{E})$. As $J(E) \otimes_{\theta} \hat{\theta}/J(\theta)\hat{E}$ is a nilpotent ideal in $\hat{E}/J(\theta)\hat{E}$, we moreover have that $J(E) \otimes_{\theta} \hat{\theta} \subseteq J(\hat{E})$. But

(1) $$\hat{E}/J(E) \otimes_{\theta} \hat{\theta} \simeq \hat{\theta}/J(\theta)\hat{\theta}$$

which is a local ring. As $J(E) \otimes_{\theta} \hat{\theta} \subseteq J(\hat{E})$, so is

(2) $$\hat{E}/J(\hat{E}) \simeq \hat{\theta}/J(\hat{\theta})$$

therefore, and thus the unity of $\hat{E}$ is primitive by Theorem I.11.2, which shows that $M \otimes_{\theta} \hat{\theta}$ is indecomposable.

For a discussion of this and the converse, which is not always true, see Huppert (1975).

The next question is: How do vertices behave under extensions? Very well, indeed:

Lemma 2. Same notation as above. Let $M \in \mathbf{M}_{\theta}(G)$ be indecomposable, and set $M\hat{} = M \otimes_{\theta} \hat{\theta}$. Then any indecomposable summand of $M\hat{}$ has the same vertices as M.

Proof: (Erdmann (1977a), p.678). Let V be a vertex of M, and let X be an indecomposable direct summand of $M\hat{}$. As inducing and tensoring with $\hat{\theta}$ commute, X is V-projective. Suppose X is W-projective for some $W \leq V$. Then the identity on X, e, considered as an element of $(M\hat{},M\hat{})^G \simeq (M,M)^G \otimes_{\theta} \hat{\theta} = (\hat{M},M)^G$ belongs to $(M\hat{},M\hat{})^G_W$, say $e = Tr^G_W(\alpha)$ for $\alpha \in (M\hat{},M\hat{})^W$. So $\alpha = \Sigma r_i\alpha_i$ for $\alpha_i \in (M,M)^W$ and $r_i \in \hat{\theta}$. Therefore

(3) $$e \in (M,M)^G_W \otimes_{\theta} \hat{\theta} \subseteq (\hat{M,M})^G.$$

However, $(M,M)^G_W \subseteq J((M,M)^G)$ as $(M,M)^G$ is local, and M is not W-projective. This is a contradiction, as e is an idempotent.

Finally, block idempotents behave just as well.

Lemma 3. Same notation as above. Let $e \in \theta[G]$ be a block idempotent and let $e = \Sigma e_i$ be a decomposition of e into a sum of block idempotents in $\hat{\theta}[G]$. Then the defect groups of e_i are those of e.

Proof: Exercise. (One possibility is to use Lemma 2 and Corollary III.5.3.)

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